THE ISKOVSKIH THEOREM FOR REGULAR SURFACES OVER IMPERFECT FIELDS

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Abstract. We generalize Iskovskih's theorem about surfaces without irregularity and bigenus from the smooth case to regular surfaces over arbitrary fields, with special focus on the case of imperfect fields. This includes surfaces that are geometrically non-normal or geometrically non-reduced. Here the usual approach of Galois descent breaks down, and one relies entirely on the scheme theory over the ground field. Moreover, the degrees of closed points can be larger than expected, and certain curves might have purely inseparable constant field extension. To deal with the latter we establish a general theory for inseparable pencils, which is of independent interest. A crucial case not present in the classical proof for Iskovskih's theorem leads to non-normal quartic surfaces that are singular along a twisted cubic, or more exotic space curves of degree three.

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Introduction

A cornerstone in the Enriques Classification of algebraic surfaces is *Castelnuovo's Rationality Criterion*: A smooth surface S over an algebraically closed ground field k is birational to $\mathbb{P}^2$ if and only if $h^1(\mathscr{O}_S) = 0$ and $h^0(\omega_S^{\otimes 2}) = 0$. Note that $H^1(S, \mathscr{O}_S)$ is the Lie algebra for the Picard scheme $\mathrm{Pic}_{S/k}$, whose dimension $h^1(\mathscr{O}_S)$ is classically

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called the *irregularity*. The invariants $h^0(\omega^{\otimes n})$ stemming from the dualizing sheaf $\omega_S = \det(\Omega^1_{S/k})$ are the *plurigenera*, and $h^0(\omega_S^{\otimes 2})$ might be called *bigenus*.

The above geometric statement can be seen as a consequence of *Iskovskih's Theorem* [24], which is of arithmetic nature: Let S be a *minimal smooth* surface with $h^1(\mathscr{O}_S) = h^2(\omega_S^{\otimes 2}) = 0$, over an *arbitrary* ground field k. Then S is isomorphic to the projective plane or a quadric surface, or there is a fibration where the base and the generic fiber are Brauer–Severi curves, or the dualizing sheaf ω_S generates the Picard group $\mathrm{Pic}(S)$. This builds on previous work of Manin [32] over perfect fields k.

The goal of this paper is to generalize Iskovskih's Theorem, to allow *minimal regular* surface over arbitrary ground fields, with special focus on the case of imperfect ground fields, similar in style to the investigations in [15]. Note that this includes surfaces that are *geometrically non-normal* or even *geometrically non-reduced*. Our main result is:

Theorem A. *(see Thm. 3.1) Let X be a minimal regular surface over an arbitrary field F, with numerical invariants $h^0(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = h^0(\omega_X^{\otimes 2}) = 0$. Then one of the following holds:*

- (i) *The surface X is isomorphic to the projective plane $\mathbb{P}^2$, or isomorphic to a quadric surface in $\mathbb{P}^3$.*
- (ii) *There is a morphism $f : X \to B$ with $f_*(\mathscr{O}_X) = \mathscr{O}_B$ such that the base and the generic fibers are regular genus-zero curves.*
- (iii) *The dualizing sheaf ω_X generates the Picard group* $\mathrm{Pic}(X)$.

In the first case of this trichotomy, the surface X can be expressed in terms of a single equation. In the second case, the geometry is reduced to dimension one. In marked contrast, it is very difficult to say more about the third case (compare [8], Section 4).

In recent years, regular del Pezzo surface over imperfect have received increased attention, for their own sake, and because they appear as generic fibers in the outputs of the Minimal Model Program for algebraic varieties in characteristic $p > 0$, for example in the work of Bernasconi, Ji, Maddock, Martin, Patakfalvi, Tanaka, and ourselves ([41], [31] [46] [15], [6], [47], [26], [37], [48], [7], [5]).

Kollár, Smith and Corti gave a highly readable presentation of Iskovskih's arguments ([28], Chapter 3), and we follow their line of reasoning: Assuming that the dualizing shed does not generate the Picard group, one produces a curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and $C^2 \geq 0$, such that C and all linearly equivalent curves are integral. *Now two major new issues arise : The field of constants $E = H^0(C, \mathscr{O}_C)$ could be a purely inseparable extension, and the self-intersection number $C^2 \geq 0$ is potentially larger that in the classical situation.* The first crucial step is to establish $h^0(\mathscr{O}_C) = 1$. Having this, we infer $C^2 \mid 4$ or $C^2 = 0$, and the cases $0 \leq C^2 \leq 3$ than correspond to (i)–(iii) in Theorem A. The second crucial step is to establish that $C^2 = 4$ actually does not appear.

To tackle the problem of constant field extensions, we develop a *general theory of purely inseparable pencils*. Suppose X is a proper normal scheme with $h^0(\mathscr{O}_X) = 1$. Let D_0, D_1 be two effective Cartier divisor stemming from global sections of some invertible sheaf $\mathscr{L}$, such that $Z = D_0 \cap D_1$ is reduced and of codimension two.

The blowing-up $V = \mathrm{Bl}_Z(X)$ comes with a fibration $h : V \to \mathbb{P}^1$, and we write $B = \operatorname{Spec} h_*(\mathscr{O}_V)$ for the Stein factorization. Our second main result reveals that the prime $p = 2$ plays a particular role:

Theorem B. *(see Thm. 5.2) In the above setting, suppose also that the cohomology group $H^1(X, \mathscr{O}_X)$ vanishes, and that for each rational point $t \in \mathbb{P}^1$, the resulting effective Cartier divisor $D_t \subset X$ is reduced and geometrically connected. The the following holds:*

(i) *The Stein factorization B is a regular genus-zero curve.*
(ii) *The map $g : B \to \mathbb{P}^1$ is a universal homeomorphism, with $\deg(g) \mid 2$.*
(iii) *For each non-zero $s \in H^0(X, \mathscr{L})$ the resulting $D \subset X$ has $h^0(\mathscr{O}_D) = \deg(g)$.*
(iv) *If $\deg(g) = 2$ then the ground field F is imperfect of characteristic $p = 2$, and B is a twisted line or a twisted ribbon.*

Here *twisted lines* and *twisted ribbons* designate regular genus-zero curves without rational points that become, after ground field extension, isomorphic $\mathbb{P}^1$ and the infinitesimal thickening $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, respectively.

Now back to our minimal regular surface X without irregularity and bigenus. To tackle the problem of self-intersection numbers, we now assume that there is an integral curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and $C^2 = 4$. The following *bend-and-break type statement* is our third main result:

Theorem C. *(see Thm. 5.2) In the above situation, the curve $C \subset X$ is linearly equivalent to a curve C' that is not integral.*

For this we examine four-dimensional linear systems for $\mathscr{L} = \mathscr{O}_X(C)$ to produce a morphism $f : X \to \mathbb{P}^3$ with $\mathscr{L} = f^*\mathscr{O}_{\mathbb{P}^3}(1)$. Using the theory of quadric hypersurfaces, it is not difficult to reduce to the case that the image is a *non-normal quartic surface* $V \subset \mathbb{P}^3$, and $f : X \to V$ is the minimal resolution of singularities. Note that singular quartic surfaces form a classical evergreen topic, as witnessed by the monographs of Hudson [23] and Jessop [25] from the beginning of the 19th century. For modern research, we mention the work of Urabe ([49], [50], [51]) and Catanese and Schütt ([12], [13], [14]).

It turns out that the branch curve B for the normalization map $g : Y \to V$ is a space curve of degree three inside $\mathbb{P}^3$. The relevant case is when it is integral, and there are two possibilities: Either B is a *twisted cubic*, a classical terminology that designates a copy of the projective line embedded into the projective three-space via $\mathscr{O}_{\mathbb{P}^1}(3)$. Or it is a *exotic cubic*, our ad hoc terminology to designates non-Gorenstein genus-zero curves arising as denormalizations of the projective line over a cubic field extension. The latter case can be treated via the conductor squares for the normalization of the *ramification curve* $R = g^{-1}(B)$, involving commutative algebra for local Artin rings.

For the twisted cubics B, an entirely different strategy is required, and we exploit the amazing fact that $\mathrm{Bl}_B(\mathbb{P}^3)$ is in an unexpected way a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, compare the work of Blanc and Lamy [10], Ray [39], and Sarkar [40]. This ensures that the induced map $\mathrm{Bl}_B(V) \to \mathbb{P}^2$ factors over a regular genus-zero curve $V_+(\Phi)$ defined by some quadratic polynomial $\Phi \in F[X_0, X_1, X_2]$, and X becomes a ruled surface over such curves.

To unravel the geometry of the situation, we are now forced to classify the locally free sheaves on such twisted ribbons B. This can be seen as a variant of *Grothendieck's Splitting Theorem* for the projective line (see [17] and [22]), and gives our fourth main result:

Theorem D. *(See Thm. 10.2) Up to isomorphism, the indecomposable locally free sheaves on twisted ribbons B are the $\omega_B^{\otimes a}$ and $\mathscr{F}_B \otimes \omega_B^{\otimes b}$, with exponents $a, b \in \mathbb{Z}$.*

Here ω_B is the dualizing sheaf, and $\mathscr{F}_B$ is the sheaf of rank two given by the non-split extension $0 \to \omega_B \to \mathscr{F}_B \to \mathscr{O}_B \to 0$ corresponding to $H^1(B, \omega_B) = F$. Similar results for Brauer–Severi curves where obtained by Biswas and Nagaray [9] and Novaković [35].

The paper is organized as follows: In Section 1 and 2 we collect generalities on genus-zero curves, twisted ribbons, and quadric hypersurfaces, which play an important role throughout. In Section 3 we set the stage and examine regular surfaces X without irregularity and bigenus over arbitrary ground fields F, and already formulate our generalization of Iskovskih's Theorem. The integral curves $C \subset X$ having $h^1(\mathscr{O}_C) = 0$ and $C^2 \geq 1$ such that all linearly equivalent curves are integral are examined in Section 4, and the possible values for $h^0(\mathscr{O}_C)$ and C^2 are determined. This relies on the theory of inseparable pencils, which is established in Section 5. In Section 6 we start to analyze the crucial new case $C^2 = 4$, and formulate the theorem that some linearly equivalent curve C' must be non-integral. This bend-and-break problem is quickly reduced to the situation that X is a modification of a non-normal quartic surface $V \subset \mathbb{P}^3$, subject to several arithmetic conditions on points with small residue field degree. In Section 7 we compute various invariants for the normalization maps $Y \to V$. It turns out that the branch curve $B \subset V$ is a space curve of degree three in $\mathbb{P}^3$, and the relevant case is when it is integral. In Section 8 we then see that B is either a *twisted cubic*, or is a non-Gorenstein genus-zero curve that arises via denormalization from a projective line over a cubic field extension $F \subset E$. We cope with such *exotic cubics* via the geometry of the conductor square. In Section 9 we treat the twisted cubics B by using the fact that the blowing-up $\mathrm{Bl}_B(\mathbb{P}^3)$ unexpectedly carries the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, and infer that X arises as $\mathbb{P}^1$-bundle over a genus-zero curve. To understand the geometry of such bundles, we generalize Grothendieck's Splitting Theorem to sheaves on regular genus-zero curves in Section 10. The final Section 11 completes the proofs of our main results.

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1. Genus-zero curves

In this section we collect some generalities that will be used later. Throughout, we fix an arbitrary ground field F of characteristic $p \geq 0$. The terms *curve* and *surfaces* refer to proper schemes that are equi-dimensional, of dimension $n = 1$ and $n = 2$, respectively. Each curve C comes with the cohomological invariants

$$h^0(\mathscr{O}_C) = \dim_F H^0(C, \mathscr{O}_C) \quad \text{and} \quad h^1(\mathscr{O}_C) = \dim_F H^1(C, \mathscr{O}_C).$$

The former is the degree of the *ring of constants* $E = H^0(C, \mathscr{O}_C)$, which is a field provided that C is reduced and connected. In any case, the integer $g = h^1(\mathscr{O}_C)$ will be called the *genus*. Note that if $\tilde{C}$ is a *twisted form* of some curve C, in other words $\tilde{C} \otimes F' \simeq C \otimes F'$ for some field extension $F \subset F'$, then $h^i(\mathscr{O}_{\tilde{C}}) = h^i(\mathscr{O}_C)$.

Following Bayer and Eisenbud [3], we say that a *ribbon* on a curve C_0 is a pair (C, ι), where $\iota : C_0 \to C$ is a closed embedding having a square-zero sheaf of ideals $\mathscr{I}$ that is invertible as $\mathscr{O}_{C_0}$-module. The ribbon *splits* if the inclusion $\iota : C_0 \to C$ admits a retraction $\rho : C \to C_0$; in this case we may regard $\mathscr{O}_C = \mathscr{O}_{C_0} \oplus \epsilon\mathscr{L}$ as a *sheaf of dual numbers*, for some invertible $\mathscr{O}_{C_0}$-module $\mathscr{L}$, and write $C = C_0 \oplus \mathscr{L}$.

Let $f : C' \to C$ be a birational morphism between curves without embedded components. Then the sheaf of conductor ideals $\mathscr{I} \subset \mathscr{O}_C$, defined as the annihilator for $f_*(\mathscr{O}_{C'})/\mathscr{O}_C$, is also a sheaf of ideals in $\mathscr{O}_{C'}$, and thus defines both the branch scheme $B \subset C$ and the ramification scheme $R \subset C'$. This yields a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & C' \arrow[d, "f"] \\ B \arrow[r] & C, \end{tikzcd} \tag{1}$$

which is both cartesian and cocartesian (see [15], Appendix A). In turn, we have a short exact sequence $0 \to \mathscr{O}_C \to \mathscr{O}_{C'} \times \mathscr{O}_B \to \mathscr{O}_R \to 0$, giving a long exact sequence

$$0 \to H^0(\mathscr{O}_C) \to H^0(\mathscr{O}_{C'}) \times H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_R) \to H^1(\mathscr{O}_C) \to H^1(\mathscr{O}_{C'}) \to 0 \tag{2}$$

Loosely speaking, the topological space $|C|$ is obtained from $|C'|$ by identifying the points in R with the same image in B, and the structure sheaf $\mathscr{O}_C$ is the kernel for $\mathscr{O}_{C'} \times \mathscr{O}_B \to \mathscr{O}_R$. The diagram (1) is called *conductor square*, and the passage from C' to C is called *pinching*.

In what follows, we are interested in the case where the invariants $h^i(\mathscr{O}_C)$ take the minimal possible values, and find the following locution useful:

Definition 1.1. A curve C with invariants $h^0(\mathscr{O}_C) = 1$ and $h^1(\mathscr{O}_C) = 0$ is called a *genus-zero curve*.

Note that for each genus-zero curve C, there are no embedded components, and the Picard scheme $\operatorname{Pic}_{C/F}$ is étale. Moreover, each twisted form of C is a genus-zero curve, and each connected modification C' of a reduced subcurve in C is a genus-zero curve, albeit over the field extension $F' = H^0(C', \mathscr{O}_{C'})$.

Each finite ring extension $F \subset E$ yields a genus-zero curve via the cocartesian

$$\begin{tikzcd} \operatorname{Spec}(E) \arrow[r] \arrow[d] & \mathbb{P}^1_R \arrow[d] \\ \operatorname{Spec}(F) \arrow[r] & C, \end{tikzcd} \tag{3}$$

and see from the exact sequence (2) that the pinching C is a genus-zero curve. A case of particular interest arises when E is a quadratic or cubic field extension.

To a large extend, the structure of genus-zero curves is determined by the following result:

Proposition 1.2. *For each irreducible genus-zero curve C the following holds:*

(i) *We have $C \simeq \mathbb{P}^1$ if and only if C admits an invertible sheaf of degree one.*
(ii) *If C is regular, the curve is a twisted form of $\mathbb{P}^1$ or $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.*
(iii) *If C is integral, Gorenstein and singular, the curve is obtained from the projective line over some quadratic extension $F \subset E$ via the pushout (3).*

In all three cases, C is a quadric plane curve. In (ii) the curve must be a twisted form of $\mathbb{P}^1$ provided $p \neq 2$. In (iii), the curve C is a twisted form of the union $\mathbb{P}^1 \cup \mathbb{P}^1$ with $\mathbb{P}^1 \cap \mathbb{P}^1 = \operatorname{Spec}(F)$, or a twisted form of $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.

Proof. Suppose there is an invertible sheaf $\mathscr{L}$ of degree one. Then $h^0(\mathscr{L}) \geq \chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathscr{O}_X) = 2$. Choose a global section $s \neq 0$, and let $Z \subset C$ be the zero scheme. The resulting short exact sequence $0 \to \mathscr{O}_C \to \mathscr{L} \to \mathscr{L}_C \to 0$ yields an exact sequence

$$H^0(C, \mathscr{L}) \to H^0(Z, \mathscr{L}_Z) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^1(X, \mathscr{L}) \longrightarrow 0.$$

It follows that $\mathscr{L}$ is globally generated and $h^1(\mathscr{L}) = 0$, and thus $h^0(\mathscr{L}) = \chi(\mathscr{L}) = 2$. The resulting morphism $f : C \to \mathbb{P}^1$ has degree one. The cokernel for $\mathscr{O}_{\mathbb{P}^1} \to f_*(\mathscr{O}_C)$ is zero-dimensional with trivial Euler characteristic, hence f is an isomorphism. This gives (i).

If C is integral and Gorenstein, the dualizing sheaf has $\deg(\omega_C) = -2\chi(\mathscr{O}_C) = -2$. Arguing as before, we infer that $\mathscr{L} = \omega_C^{\otimes -1}$ is globally generated with $h^0(\mathscr{L}) = 3$, and the resulting morphism $f : C \to \mathbb{P}^2$ is a closed embedding, with image of degree two. If C is geometrically reduced, we use (i) after base-changing to F^{alg} and see that C is a twisted form of $\mathbb{P}^1$. If C is geometrically non-reduced, we regard it as a quadric curve in $\mathbb{P}^2$, and see that it is a twisted form of $2L \subset \mathbb{P}^2$ for some line L. This is a ribbon on $\mathbb{P}^1$ with sheaf of ideals $\mathscr{I} = \mathscr{O}_{\mathbb{P}^1}(-1)$. From $\operatorname{Ext}^1(\Omega^1_{\mathbb{P}^1}, \mathscr{I}) = H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2-1)) = 0$ we see that the ribbons splits, so C is a twisted form of $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. This establishes (ii).

Suppose that C is integral, Gorenstein and singular. Let $C' \to C$ be the normalization, and form the conductor square (1). The C' is a regular genus-zero curve over the field extension $F' = H^0(C', \mathscr{O}_{C'})$. Set

$$d = [F' : F] \quad \text{and} \quad l = \dim_{F'} H^0(R, \mathscr{O}_R) \quad \text{and} \quad \lambda = dl - \dim_F H^0(B, \mathscr{O}_B).$$

The short exact sequence (2) yields $1 - (d + (dl - \lambda)) + \lambda$. The Gorenstein condition ensures $\lambda = dl/2$, see [15], Proposition A.2, and the equation becomes $d(l-2) = 2$. The only solution is $d = 2$ and $l = 1$, which gives (iii).

The remaining statement arise as follows: From Proposition 2.1 below we see that in characteristic $p \neq 2$ are regular quadric curve $C \subset \mathbb{P}^2$ must be smooth, hence are twisted forms of $\mathbb{P}^1$. If C is as in (iii), then the tensor product $E \otimes F^{\text{alg}}$ is isomorphic to the product algebra $F^{\text{alg}} \times F^{\text{alg}}$ or the ring of dual numbers $F^{\text{alg}}[\epsilon]$, producing the union of lines or the ribbon, respectively. $\square$

Corollary 1.3. *Let C be a genus-zero curve that is integral, Gorenstein, and contains at most one rational point. Then the closed points $c \in C$ of degree two are Zariski dense, and their local rings $\mathscr{O}_{C,c}$ are regular. If C is singular, each $\mathscr{O}_C(c)$ generates the Picard group.*

Proof. By the proposition, there is a globally generated invertible sheaf $\mathscr{L}$ of degree two. Choose two global sections s_0, s_1 that generate $\mathscr{L}$. The resulting morphism $f : C \to \mathbb{P}^1$ has degree two. For each rational point $t \in \mathbb{P}^1$, the preimage is either a degree two point, or contains a rational point. Thus the degree two points $c \in C$ are Zariski dense.

If C is singular, its description as pinching from $\mathbb{P}^1_E$ shows that the singular locus comprises a unique point that is rational, so the $\mathscr{O}_{C,c}$ must be regular. Since the degree map $\deg : \operatorname{Pic}(C) \to \mathbb{Z}$ is injective and there is no invertible sheaf of degree one, each $\mathscr{O}_C(c)$ is a generator of the Picard group. $\square$

The following terminology will be useful throughout: The twisted forms of the projective line are called *Brauer–Severi curves*, and those having no rational points are referred to as *twisted lines*. The twisted forms of $\mathbb{P}^1 \cup \mathbb{P}^1$ with $\mathbb{P}^1 \cap \mathbb{P}^1 = \operatorname{Spec}(F)$ that are integral are called *twisted line pairs*. If C is a twisted form of $C_0 = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$ that is regular, we say that C is a *twisted ribbons*; if it is merely reduced and singular we call it a *twisted ribbon with singularity*. Note that this is only possible over imperfect fields F in character $p = 2$, and the former actually needs $\operatorname{pdeg}(F) \geq 2$.

Corollary 1.4. *Let C be a twisted ribbon, and $F \subset F'$ be an inseparable quadratic extension. Then the base-change $C' = C \otimes F'$ is a twisted ribbon, or a twisted ribbon with singularity. The latter holds if and only there is a closed point $c \in C$ with $\kappa(c) \simeq F'$.*

Proof. First note that $p = 2$, and that the base change C' is a twisted form of $\mathbb{P}^1_{F'} \oplus \mathscr{O}_{\mathbb{P}^1_{F'}}(-1)$. It must be integral by [43], Lemma 1.3. It regular, it is a twisted ribbon. If singular, it must be a twisted ribbon with singularity, according to the proposition. $\square$

Lemma 1.5. *Let C be a twisted ribbon, and $F \subset E$ be an inseparable quadratic extension. Then there is no morphisms $f : \mathbb{P}^1_E \to C$ of degree two.*

Proof. Suppose such a morphism exists. Then f is flat, and we get a short exact sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow f_*(\mathscr{O}_{\mathbb{P}^1_E}) \longrightarrow \mathscr{L} \longrightarrow 0$$

for some invertible sheaf $\mathscr{L}$ on the twisted ribbon C. From $\chi(\mathscr{O}_C) = 1$ and $\chi(\mathscr{O}_{\mathbb{P}^1_E}) = 2$ we get $\deg(\mathscr{L}) = \chi(\mathscr{L}) - \chi(\mathscr{O}_C) = 0$, and thus $\mathscr{L} = \mathscr{O}_C$. Using $\operatorname{Ext}^1(\mathscr{O}_C, \mathscr{O}_C) = H^1(C, \mathscr{O}_C) = 0$, we see that the above short exact sequence of $\mathscr{O}_C$-modules splits. In turn, the ring structure takes the form $f_*(\mathscr{O}_{\mathbb{P}^1_E}) = \mathscr{O}_C[T]/(P)$ for some polynomial $P(T) = T^2 + \alpha T + \beta$ whose coefficients belong to $H^0(C, \mathscr{O}_C) = k$. It follows that $\mathbb{P}^1_E = C \otimes_F E'$ for some quadratic field extension $F \subset E'$. Taking global sections we obtain $E' = E$. By Proposition 1.4, the base change $C \otimes E$ is a twisted ribbon, perhaps with singularity, By Proposition 1.2 it contains at most one rational point, contradiction. □

Lemma 1.6. *Let $f : C' \to C$ be a finite flat morphism of degree two between genus-zero curves. If C is integral, it must be isomorphic to $\mathbb{P}^1$.*

Proof. By flatness, the cokernel $\mathscr{L} = f_*(\mathscr{O}'_C)/\mathscr{O}_C$ is invertible, with Euler characteristic $\chi(\mathscr{L}) = \chi(\mathscr{O}_{C'}) - \chi(\mathscr{O}_C) = 0$. Then $\deg(\mathscr{L}) = \chi(\mathscr{L}) - \chi(\mathscr{O}_C) = -1$, and the assertion follows from Proposition 1.2. □

2. Quadric hypersurfaces

We keep the set-up from the previous section, and now examine *quadric hypersurfaces* $V \subset \mathbb{P}^{n+1}$ of arbitrary dimension $n \geq 0$. They are defined by a non-zero homogeneous polynomial Φ of degree two in $n+2$ indeterminates with coefficients from F. After a change of coordinates, we may write

$$\Phi = \sum_{i=0}^{m} \lambda_i T_i^2 \tag{4}$$

in the indeterminates T_i, for some $0 \leq m \leq n+1$ and $\lambda_i \in F^\times$, provided $p \neq 2$. In characteristic two, we instead may write

$$\Phi = \sum_{i=1}^{r}(X_iY_i + \alpha_i X_i^2 + \beta_i Y_i^2) + \sum_{j=1}^{s} \gamma_j Z_j^2 \tag{5}$$

in some indeterminates X_i, Y_i, Z_j, with $0 \leq 2r + s - 1 \leq n+1$ and $\alpha_i, \beta_i, \gamma_j \in F$, where the $\gamma_1, \ldots, \gamma_s$ are linearly independent over the subfield F^p, and $r = 0$ or $s = 0$ means that the respective sums disappear. See [1], Satz 2 for more details.

Note that our designations T_i or X_i, Y_i, Z_j for the indeterminates will indicates that the characteristic is $p \neq 2$ or $p = 2$, respectively. In the latter case, let $F^p \subset E$ be the height-one extension generated by the fractions γ_j/γ_k, $1 \leq j, k \leq s$, and consider the ensuing number

$$\delta_{\text{reg}} = \text{pdeg}(E/F^p).$$

Recall that the *p-degree* is the cardinality of a p-basis of E over F^p, or equivalently of a basis for Ω^1_{E/F^p} over E. The *jacobian ideal* $\mathfrak{j} = (X_1, Y_1, \ldots, X_r, Y_r)$ defines a linear subspace $\mathbb{P}^{s-1}$; for $s \geq 1$ we write $V_0 \subset \mathbb{P}^{s-1}$ for the induced quadric hypersurface, which is defined by the the Fermat polynomial $\Phi_0 = \sum_{j=1}^{s} \gamma_j Z_j^2$.

Proposition 2.1. *In the above situation, the scheme of non-smoothness* $\text{Sing}(V/F)$ *is the subscheme in $\mathbb{P}^{n+1}$ defined by the respective homogeneous ideals*

$$\mathfrak{a} = (T_0, \ldots, T_m) \quad \textit{and} \quad \mathfrak{b} = (X_1, Y_1, \ldots, X_r, Y_r, \gamma_1 Z_1^2 + \ldots + \gamma_s Z_s^2).$$

This coincides with the singular locus $\text{Sing}(V)$*, except for $p = 2$ and $s \geq 1$. In this case,* $\text{Sing}(V) \subset V$ *has codimension $2r + \delta_{\text{reg}}$, and coincides with the singular locus of the Fermat hypersurface $V_0 \subset \mathbb{P}^{s-1}$.*

Proof. Computing partial derivatives, one immediately gets the statement on the scheme of non-smoothness. The second statement is a special case of [43], Theorem 3.3. □

For $p \neq 2$ we thus see that $\mathrm{Sing}(V/F) = \mathrm{Sing}(V)$. If non-empty, this locus has codimension $0 \leq m \leq n$ inside the n-dimensional quadric hypersurface V. In characteristic two, we get

$$\mathrm{codim}_V \mathrm{Sing}(V/F) = 2r \quad \text{and} \quad \mathrm{codim}_V \mathrm{Sing}(V) = 2r + \delta_{\mathrm{reg}}.$$

Using Serre's Criterion for normality ([20], Theorem 5.8.6), we immediately get:

Corollary 2.2. *The quadric hypersurface $V \subset \mathbb{P}^{n+1}$ is reduced if and only if $m \geq 1$ or $r \geq 1$ or $\delta_{\mathrm{reg}} \geq 1$. It is normal if and only if $m \geq 2$ or $r \geq 1$ or $\delta_{\mathrm{reg}} \geq 2$.*

The following observations are useful throughout:

Lemma 2.3. *The singular locus $\mathrm{Sing}(V)$ is connected. If V is reduced and non-normal, the normalization is isomorphic to $\mathbb{P}^n_E$, where $F \subset E$ is either a quadratic field extension or the product ring $E = F \times F$.*

Proof. The connectedness statement immediately follows from Proposition 2.1, except in the case $p = 2$ and $s \geq 1$. It then suffices to treat the case

$$\Phi = \sum_{j=1}^{s} \gamma_j Z_j^2 \quad \text{and} \quad s = n+2 \quad \text{and} \quad r = 0.$$

Set $t = \delta_{\mathrm{reg}}$. Without loss of generality we may assume that $\gamma_1, \ldots, \gamma_t \in F$ are p-linearly independent over F^p, and that $\gamma_{t+1} = 1$. By [11], Chapter V, §13, No. 2, Theorem 1 there are derivations $D_i : F \to F$ with $D_i(\gamma_j) = \delta_{ij}$ for $1 \leq i, j \leq t$, and thus

$$D_i(\Phi) = Z_i^2 + \sum_{j=t+2}^{s} D_i(\gamma_j) Z_j^2.$$

Set $\mathfrak{c} = (\Phi, D_1(\Phi), \ldots, D_t(\Phi))$. The resulting closed subscheme $V_+(\mathfrak{c}) \subset \mathbb{P}^{n+1}$ is isomorphic to the hypersurface in $\mathrm{Proj}\, F[Z_{t+1}, \ldots, Z_{n+2}]$ defined by a Fermat equation of the form $Z_{t+1}^2 + \sum_{j=t+2}^{s} \gamma_j' Z_j^2 = 0$, hence is irreducible of dimension $(n+2) - (t+1) = n + 1 - \delta_{\mathrm{reg}}$. It contains $\mathrm{Sing}(V)$, which by Proposition 2.1 has the same dimension. Thus the inclusion $\mathrm{Sing}(V) \subset V_+(\mathfrak{c})$ is an equality, and connectedness follows.

Now suppose that V is reduced and non-normal, with normalization V'. Assume first $p \neq 2$. By Corollary 2.2, it suffices to treat the case $\Phi = T_0^2 - \lambda T_1^2$ for some $\lambda \in F^\times$, which in characteristic two does not belong to $F^{2\times}$. Then the ring $E = F[U]/(U^2 - \lambda)$ is regular, and we write $\omega \in E$ for the class of the indeterminate U. As for [43], Proposition 4.1, one easily checks that the homomorphism

$$F[T_0, \ldots, T_{n+1}] \longrightarrow E[T_1', \ldots, T_{n+1}']$$

of graded F-algebras given by $T_0 \mapsto \omega T_1'$ and $T_i \mapsto T_i'$ for $1 \leq i \leq n+1$ induces a birational morphism $\mathbb{P}^n_E \to V \subset \mathbb{P}^{n+1}$, hence $V' = \mathbb{P}^n_E$. □

3. Surfaces without irregularity and bigenus

Let F be a ground field of characteristic $p \geq 0$, not necessarily perfect, and X be a regular surface, not necessarily smooth, with numerical invariants

$$h^0(\mathscr{O}_X) = 1 \quad \text{and} \quad h^1(\mathscr{O}_X) = h^0(\omega_X^{\otimes 2}) = 0.$$

Clearly, we have $h^0(\omega_X) = 0$, and Serre Duality ensures $h^2(\mathscr{O}_X) = 0$, such that $\chi(\mathscr{O}_X) = 1$. The Chern number $c_1^2 = (\omega_X \cdot \omega_X)$ and the Picard number $\rho \geq 1$ are further invariants of interest. The same goes for invariants stemming from the coherent sheaf $\Omega^1_{X/F}$, which is locally free of rank two if and only if X is smooth.

Throughout we also assume that X is *minimal*. In other words, there is no integral curve $R \subset X$ such that

$$(R \cdot R) = (R \cdot \omega_X) = -h^0(\mathscr{O}_E).$$

Such R are isomorphic to the projective line over $E = H^0(R, \mathscr{O}_R)$, and are called *exceptional curves of the first kind*; one may call them (-1)-*curves* as well, in reference to the sign occurring the above equations. We can already state the main result of this paper:

Theorem 3.1. *In the above situation, one of the following conditions holds:*

(i) *The surface X is isomorphic to $\mathbb{P}^2$, or to a quadric in $\mathbb{P}^3$.*
(ii) *There is a morphism $f : X \to B$ with $f_*(\mathscr{O}_X) = \mathscr{O}_B$ such that the base and the generic fiber are regular genus-zero curves.*
(iii) *The dualizing sheaf ω_X generates the Picard group* $\operatorname{Pic}(X)$.

This trichotomy generalizes Iskovskih's result ([24], Theorem 1), by replacing smoothness with regularity and allowing imperfect base fields. The proof requires extensive preparation, and will be completed in Section 11. Note that our surface X may be geometrically non-normal, or even geometrically non-reduced. In this general setting, one has to cope with two new issues: The integral curves $C \subset X$ may have larger self-intersection C^2, compared to the smooth case, and the field of constants $E = H^0(C, \mathscr{O}_C)$ could be an extension of the ground field F.

In this section we closely follow the exposition of Kollár, Smith and Corti ([28], Section 3.2), but with special attention to the above issues. Let us start with the following key fact, classically called "Termination of Adjunction".

Lemma 3.2. *For each invertible sheaf $\mathscr{L}$ we have $h^0(\mathscr{L} \otimes \omega_X^{\otimes n}) = 0$ for all sufficiently large $n \geq 0$.*

Proof. Serre Duality ensures $h^2(\omega_X^{\otimes -1}) = h^0(\omega_X^{\otimes 2})$, which vanishes by our assumption. Thus Riemann–Roch gives $h^0(\omega_X^{\otimes -1}) \geq \chi(\omega_X^{\otimes -1}) = c_1^2 + \chi(\mathscr{O}_X) = c_1^2 + 1$. We have $\omega_X^{\otimes -1} \not\simeq \mathscr{O}_X$ because $h^0(\omega_X) = 0$. So if $c_1^2 \geq 0$, then each ample divisor $H \subset X$ has $(\omega_X \cdot H) < 0$, therefore $(\mathscr{L} \otimes \omega_X^{\otimes n} \cdot H) < 0$ for all n sufficiently large, and then $h^0(\mathscr{L} \otimes \omega_X^{\otimes n}) = 0$.

Suppose now $c_1^2 < 0$. Then $(\mathscr{L} \otimes \omega_X^{\otimes n} \cdot \omega_X) < 0$ for all $n \geq n'$ with some integer n', and then the $\mathscr{L} \otimes \omega_X^{\otimes n}$ are non-trivial. Seeking a contradiction, we assume that for infinitely many $n_i \geq 0$, $i \geq 0$ the invertible sheaf $\mathscr{L} \otimes \omega_X^{\otimes n_i}$ admits a non-zero global section s_i. Without loss of generality we may assume $n_i \geq n'$. Let $D_i \subset X$ be the zero scheme of the s_i. Decompose $D_0 = \sum m_j C_j$ into irreducible components.

Then some $C = C_j$ has $(C \cdot \omega_X) < 0$. Since X is minimal, we must have $C^2 \geq 0$, and thus $(\mathscr{L} \otimes \omega_X^{\otimes n_i} \cdot C) \geq 0$ for all $i \geq 0$. On the other hand, $(\omega_X \cdot C) < 0$ ensures that for sufficiently large $i \geq 0$ we have $(\mathscr{L} \otimes \omega_X^{\otimes n_i} \cdot C) < 0$, contradiction. □

This has an immediate consequence for the Picard scheme:

Proposition 3.3. *The canonical map* $\mathrm{Pic}_{X/k} \to \mathrm{Num}_{X/k}$ *of group schemes is an isomorphism.*

Proof. The Lie algebra for $\mathrm{Pic}_{X/k}$ is the cohomology group $H^1(X, \mathscr{O}_X)$. This vanishes by assumption, so $\mathrm{Pic}^0_{X/k}$ is trivial. To see that the local system $\mathrm{Num}_{X/k}$ contains no torsion, it suffices to treat the case that k is separably closed. Seeking a contradiction, we assume that there is an invertible sheaf $\mathscr{L}$ of finite order $m > 1$ in the Picard group. Then $\mathscr{L}$ is numerically trivial with $h^0(\mathscr{L}) = 0$, so Serre Duality and Riemann–Roch gives $h^2(\mathscr{L}) \geq \chi(\mathscr{L}) = \chi(\mathscr{O}_X) = 1$. Again by Serre duality, the invertible sheaf $\mathscr{N} = \omega_X \otimes \mathscr{L}^\vee$ admits a non-zero global section, and the same holds for the tensor power $\mathscr{N}^{\otimes m} = \omega_X^{\otimes m}$. Thus $h^0(\omega_X^{\otimes n}) \neq 0$ for all positive multiples n of m, in contradiction to Lemma 3.2. □

We conclude that $\mathrm{Pic}(X) = \mathrm{Num}(X)$ is a free abelian group. Furthermore, the class of the dualizing sheaf ω_X is non-zero, in light of Lemma 3.2.

Proposition 3.4. *Suppose the dualizing sheaf ω_X does not generate the Picard group* $\mathrm{Pic}(X)$*. Then there is an integral curve $C \subset X$ such that $h^1(\mathscr{O}_C) = 0$ and $C^2 \geq 0$, and that every linearly equivalent $C' \subset X$ is also integral.*

Proof. For each ample divisor $H \subset X$, we consider the function $n \mapsto h^0(\omega_X^{\otimes n}(H))$. It is non-zero for $n = 0$, but vanishes for n sufficiently large, by Lemma 3.2. Let $n = n_H$ be the largest integer such that $\mathscr{N} = \omega_X^{\otimes n}(H)$ has a non-zero global section but $\mathscr{N} \otimes \omega_X = \omega_X^{\otimes n+1}(H)$ does not. Fix such a global section s, and regard it as homomorphism $s : \omega_X^{\otimes -n} \to \mathscr{O}_X(H)$. Now recall that $\mathrm{Pic}(X)$ is generated by the classes of very ample divisors. If the s are bijective for all H, then ω_X would generate the Picard group, contradiction. Therefore for some H the map s is not bijective.

Summing up, we find some non-trivial invertible sheaf $\mathscr{N}$ with $h^0(\mathscr{N}) \neq 0$ but $h^0(\mathscr{N} \otimes \omega_X) = 0$. Now re-choose a non-zero $s \in H^0(X, \mathscr{N})$ so that the zero scheme $D \subset X$ maximizes $\sum_{i=1}^r m_i$, where $D = \sum_{i=1}^r m_i C_i$ is the decomposition into irreducible components. Let $C = \sum m_i' C_i$ be any subcurve of D. The resulting invertible sheaf $\mathscr{L} = \mathscr{O}_X(C)$ has the property that $h^0(\mathscr{L}^{\otimes -1}) = 0$ and $h^0(\mathscr{L} \otimes \omega_X) = 0$. Now Serre Duality ensures $h^2(\mathscr{L}^{\otimes -1}) = 0$, and Riemann–Roch gives

$$0 = h^0(\mathscr{L}^{\otimes -1}) + h^2(\mathscr{L}^{\otimes -1}) \geq \chi(\mathscr{L}^{\otimes -1}) = \frac{(C + K_X) \cdot C}{2} + \chi(\mathscr{O}_X). \tag{6}$$

The Adjunction Formula yields $\deg(\omega_C) = (C + K_X) \cdot C \leq -2\chi(\mathscr{O}_X) = -2$, whereas Serre Duality gives $\deg(\omega_C) = -2\chi(\mathscr{O}_C)$, thus $\chi(\mathscr{O}_C) \geq 1$.

Now additionally assume that for the subcurve $C \subset D$, the finite k-algebra $E = H^0(C, \mathscr{O}_C)$ is a field. The latter condition holds, for example, if C is reduced and connected, and in particular if C is integral. The cohomology group $H^1(C, \mathscr{O}_C)$ carries the structure of an E-vector space. We have

$$1 \leq \chi(\mathscr{O}_C) = [E : k](1 - \dim_E H^1(C, \mathscr{O}_C))$$

and conclude $h^1(\mathscr{O}_C)=0$. In particular, C is a genus-zero curve over the extension field E.

The preceding paragraph applies in particular to $C=C_i$, so each C_i is a genus-zero curve over the field $E_i=H^0(C_i,\mathscr{O}_{C_i})$. Seeking a contradiction, we assume now that $C_i^2<0$ for all $1\leq i\leq r$. If $C_i^2=-[E_i:k]$ then $C_i\subset X$ would be an exceptional curve of the first kind, contradiction. Thus $C_i^2\leq -2[E_i:k]$, and from

$$-2[E_i:k]=\chi(\mathscr{O}_{C_i})=\deg(\omega_{C_i})=C_i^2+(\omega_X\cdot C)\leq -2[E_i:k]+(\omega_X\cdot C),$$

we infer $(\omega_X\cdot C_i)\geq 0$, and thus $(\omega_X\cdot D)\geq 0$. Now recall that $\omega_X^{\otimes -n}(D)=\mathscr{O}_X(H)$ is ample, and therefore

$$(\omega_X(D)\cdot D)=(\omega_X^{\otimes -n}(D)\cdot D)+(\omega_X^{\otimes n+1}\cdot D)\geq(\omega_X^{\otimes n+1}\cdot D)\geq 0.$$

On the other hand, we saw in (6) with $C=D$ that $(\omega_X(D)\cdot D)<0$, contradiction. This shows that some $C=C_j$ has $h^1(\mathscr{O}_C)=0$ and $C^2\geq 0$. By maximality of $\sum_{i=1}^r m_i\geq 1$, every linearly equivalent curve C' remains integral. □

Proposition 3.5. *Let $C\subset X$ be an integral curve with $h^1(\mathscr{O}_C)=0$ and $C^2\geq 0$. Then the invertible sheaf $\mathscr{L}=\mathscr{O}_X(C)$ is globally generated, with $h^0(\mathscr{L})=1+h^0(\mathscr{O}_C)+C^2$.*

Proof. The short exact sequence $0\to\mathscr{O}_X\to\mathscr{L}\to\mathscr{L}_C\to 0$ gives a long exact sequence

$$0\longrightarrow H^0(X,\mathscr{O}_X)\longrightarrow H^0(X,\mathscr{L})\longrightarrow H^0(C,\mathscr{L}_C)\longrightarrow H^1(X,\mathscr{O}_X),$$

and the term on the right vanishes. So to see that $\mathscr{L}$ is globally generated, it suffices to verify this for $\mathscr{L}_C$. This indeed holds, because C is an integral genus-zero curve over the field $E=H^0(C,\mathscr{O}_C)$, and $\deg(\mathscr{L}_C)=C^2\geq 0$. Furthermore, we have $h^1(\mathscr{L}_C)=0$, and thus $h^0(\mathscr{L}_C)=\chi(\mathscr{L}_C)=\deg(\mathscr{L}_C)+\chi(\mathscr{O}_C)=C^2+h^0(\mathscr{O}_C)$. The assertion on $h^0(\mathscr{L})$ follows from the above long exact sequence. □

In the above situation, the globally generated invertible sheaf $\mathscr{L}=\mathscr{O}_X(C)$ defines a morphism

$$f:X\longrightarrow\mathbb{P}^n\qquad\text{with}\qquad f^*\mathscr{O}_{\mathbb{P}^n}(1)=\mathscr{L},$$

where $n=h^0(\mathscr{O}_C)+C^2$. The schematic image $V\subset\mathbb{P}^n$ is an integral closed subscheme of dimension one or two. This image is not a linear subscheme, because the induced map $H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(1))\to H^0(X,\mathscr{L})$ is bijective. Note that the image may or may not be regular, and that we have the formula

$$C^2=\deg(f)\cdot\deg(V),\tag{7}$$

where the degree for $f:X\to V$ is formed in the sense of Kleiman ([27], Chapter I, Section 2, Definition on page 299). The following already settles parts of Theorem 3.1:

Proposition 3.6. *Suppose there is an integral curve $C\subset X$ with $h^1(\mathscr{O}_C)=0$ and $0\leq C^2\leq 2$. Then one of the following holds:*

(i) *The surface X is isomorphic to the projective plane $\mathbb{P}^2$, or isomorphic to a quadric surface in $\mathbb{P}^3$.*
(ii) *There is morphism $g:X\to B$ with $\mathscr{O}_B=g_*(\mathscr{O}_X)$ such that the base B and and the generic fiber X_η are regular genus-zero curves.*

Moreover, X is geometrically normal provided the characteristic is $p \neq 2$.

Proof. Suppose first that $C^2 = 0$. Then the image of $f : X \to \mathbb{P}^n$ must be a curve. Let $B = \operatorname{Spec} f_*(\mathscr{O}_X)$ be the Stein factorization, and $g : X \to B$ be the resulting morphism. Then B is integral and normal. The Leray–Serre spectral sequence gives $H^0(B, \mathscr{O}_B) = H^0(X, \mathscr{O}_X)$ and an exact sequence

$$0 \longrightarrow H^1(B, \mathscr{O}_B) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^0(B, R^1 g_*(\mathscr{O}_X)) \longrightarrow 0.$$

Consequently B is a regular genus-zero curve and $R^1 g_*(\mathscr{O}_X)$ is locally free. Since $C^2 = 0$ the image $f(C) \subset \mathbb{P}^n$ is a singleton, and thus $C = g^{-1}(b)$ for some closed point $b \in B$. The formation of $R^1 g_*(\mathscr{O}_X)$ commutes with base-change, because g is flat with one-dimensional fibers, and thus $R^1 g_*(\mathscr{O}_X) \otimes \kappa(b) = H^1(C, \mathscr{O}_C) = 0$. Semicontinuity now gives $R^1 g_*(\mathscr{O}_X) \otimes \kappa(\eta) = 0$, in other words, the generic fiber X_η is a genus-zero curve.

For the remaining cases the image $V \subset \mathbb{P}^n$ is an integral surface, subject to (7). Also note that the integer $h^0(\mathscr{O}_C)$ divides $(\mathscr{L} \cdot C) = C^2$. Suppose now that $C^2 = 1$. Then $f : X \to V$ is birational, $h^0(\mathscr{O}_D) = 1$ and thus $h^0(\mathscr{L}) = 3$. It follows $V = \mathbb{P}^2$. This gives a birational morphism $f : X \to \mathbb{P}^2$, which is a sequence of blowing-ups. It must be an isomorphism because X is minimal.

It remains to treat the case $C^2 = 2$. We then have $1 \leq h^0(\mathscr{O}_C) \leq 2$ and thus $4 \leq h^0(\mathscr{L}) \leq 5$. To cope with the possible constant field extension, we observe that $\mathscr{L}|C$ is generated by two global sections. We thus find $s_0, \ldots, s_3 \in H^0(X, \mathscr{L})$ without common zero, and now replace f by the morphism $f : X \to \mathbb{P}^3$ with $f^{-1}(t_i) \otimes 1 = s_i$, where $\mathbb{P}^3 = \operatorname{Proj} F[t_0, \ldots, t_3]$. Now the image V must be a quadric surface, and the induced $f : X \to V$ is birational. By the argument in the preceding paragraph, f is an isomorphism provided that V is regular.

Seeking a contradiction, we assume that the quadric surface V is not regular. The morphism $f : X \to V$ factors over the normalization $V' \to V$, thus $h^0(\mathscr{O}_{V'}) = 1$. It follows from Lemma 2.3 that V is normal, and the singular locus is a singleton $v_0 \in V$. It follows from the description in Proposition 2.1 that this point is rational. Choose some global section of $\mathscr{O}_V(1)$ so that the zero scheme $D \subset V$ passes through v_0. Then the preimage $f^{-1}(D)$ is linearly equivalent to C but reducible, contradiction.

It remains to check that X is geometrically normal, provided $p \neq 2$. If $C^2 = 0$, then the base and the generic fiber of the fibration $g : X \to B$ are smooth, according to Proposition 1.2, so the $\operatorname{Sing}(X/F)$ is finite. By Serre's Criterion ([20], Theorem 5.8.6) the scheme X is geometrically normal. There is nothing to prove if $X = \mathbb{P}^2$, so it remains to treat the case that $X \subset \mathbb{P}^3$ is a regular quadric. Then X is smooth, by Proposition 2.1. □

4. Fields of constants and self-intersections

We keep the set-up from the previous section, such that X is a minimal regular surface over a ground field F of characteristic $p \geq 0$, with $h^0(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = h^0(\omega_X^{\otimes 2}) = 0$. We also assume that there is an integral curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and self-intersection $C^2 > 0$, such that all linearly equivalent curves C' are also integral. By Proposition 3.5 the invertible sheaf $\mathscr{L} = \mathscr{O}_X(C)$ is globally generated and has

$$h^0(\mathscr{L}) = 1 + h^0(\mathscr{O}_C) + C^2 \geq 3. \tag{8}$$

In this section, we will determine the second summand, and constrain the third.

Proposition 4.1. *The degree of the constant field extension is $h^0(\mathscr{O}_C) = 1$.*

Proof. Seeking a contradiction, we assume that $E = H^0(C, \mathscr{O}_C)$ is a non-trivial extension of the ground field F. Using $C^2 > 0$ and the Hodge Index Theorem ([18], Theorem 1.1), we see that C is geometrically connected, thus $F \subset E$ is purely inseparable ([11], Chapter V, §7, No. 8, Proposition 13). Hence we are in characteristic $p > 0$ and the field F is imperfect. By Theorem 5.2 below we have $p = 2$ and $[E : F] = 2$.

Regarding C as a genus-zero curve over E, we find with Corollary 1.3 a closed point $z \in C$ such that the residue field $K = \kappa(z)$ has degree at most two over E, and that the local ring $\mathscr{O}_{C,z}$ is regular. Choose a global section of $\mathscr{L}|C$ that vanishes at $z \in C$ and has only simple zeros, and extend it to a global section $s' \in H^0(X, \mathscr{L})$. Let $C' \subset X$ be its zero-scheme and set $Z = C \cap C'$. The ensuing blowing-up $r : \mathrm{Bl}_Z(X) \to X$ comes with a fibration $h : \mathrm{Bl}_Z(X) \to \mathbb{P}^1$, as explained in the next section. Again by Theorem 5.2, the Stein factorization B is a twisted line or twisted ribbon. In particular, our ground field F is infinite. Moreover, the projection $g : B \to \mathbb{P}^1$ is radical of degree two, and for each rational point $t \in \mathbb{P}^1$, the fiber $g^{-1}(t)$ is reduced.

Suppose that B is a twisted line. First note that over some dense open set the formation of $h_*(\mathscr{O}_X)$ commutes with arbitrary base-change. By Corollary 1.3 we can pick a closed point $b \in B$ that is contained in this open set and whose residue field $L = \kappa(b)$ is a separable quadratic extension. Then the image $g(b) \in \mathbb{P}^1$ is not rational, hence also has residue field L. Consider the corresponding L-valued points $b \in B \otimes_F L$ and $g(b) \in \mathbb{P}^1_L$. The base-change $B \otimes_F L$ is the projective line over L, and the fiber over $t = g(b)$ has coordinate ring $L[\epsilon]$. So this point corresponds to a divisor D on the base-change $X \otimes L$ that is linearly equivalent to $C \otimes L$ and has $H^0(D, \mathscr{O}_D) = L[\epsilon]$, and the structure morphism $D \to \operatorname{Spec} L[\epsilon]$ is flat. Since $X \otimes L$ remains regular, the subcurve $H \subset D$ defined by $\epsilon = 0$ is Cartier, with $D = 2H$, and thus $H^2 > 0$. On the other hand, the short exact sequence $0 \to \epsilon\mathscr{O}_H \to \mathscr{O}_D \to \mathscr{O}_H \to 0$ gives $H^2 = -\deg(\epsilon\mathscr{O}_H) = 0$, contradiction.

It remains to treat the case that B is a twisted ribbon. The fiber $r^{-1}(z)$ of the blowing-up is a copy of $\mathbb{P}^1_K$, and the degree of composite map

$$\mathbb{P}^1_K \longrightarrow B \longrightarrow \mathbb{P}^1$$

is the degree of the finite scheme $r^{-1}(z) \cap h^{-1}(0) = \operatorname{Spec}(K)$. By construction $[K : F] = [K : E] \cdot [E : F] \leq 4$. Using Lemma 1.5 we see $\deg(\mathbb{P}^1_K/B) > 2$, which gives $[K : F] > 4$, contradiction. $\square$

So formula (8) simplifies to $h^0(\mathscr{L}) = 2 + C^2$. It turns out that there are only three possibilities:

Proposition 4.2. *The self-intersection number satisfies $C^2 \mid 4$. If there are two rational points $a \neq b$ on X we actually have $C^2 \mid 2$.*

Proof. First suppose that there are rational points $a \neq b$, and consider the finite subscheme $Z \subset X$ with coordinate ring $\mathscr{O}_{X,a}/\mathfrak{m}_a \times \mathscr{O}_{X,b}/\mathfrak{m}_b^2$, which has $h^0(\mathscr{O}_Z) = 1 + 3 = 4$. If $C^2 \geq 3$ we have $h^0(\mathscr{L}) \geq 5$, and find some curve C' that is linearly equivalent to C and contains Z. Then C' is a genus-zero curve that is integral, Gorenstein and singular with two rational points, in contradiction to Proposition 1.2. This settles the second assertion.

Suppose now that X contains at most one rational point. By Corollary 1.3, the integer $C^2 \geq 1$ is even. Seeking a contradiction, we now assume that $C^2 \geq 6$, hence $h^0(\mathscr{L}) \geq 8$. Fix a closed point $c \in C$ of degree two and form the finite subscheme $Z \subset X$ with coordinate ring $\mathscr{O}_{X,c}/\mathfrak{m}_c^2$, which has $h^0(Z) = 6$. Again we find a curve C' that is linearly equivalent to C and contains Z. By construction, the local ring $\mathscr{O}_{C',c}$ is singular and the residue field $\kappa(c)$ has degree two, contradiction. $\square$

5. Inseparable pencils

In this section we establish some general facts, which appear to be of independent interest and have been used in the previous section. Let F be a ground field of characteristic $p \geq 0$, and X be a proper normal scheme with $h^0(\mathscr{O}_X) = 1$, of dimension $n \geq 2$. Let $\mathscr{L}$ be an invertible sheaf not isomorphic to $\mathscr{O}_X$. For each non-zero global section $s \in H^0(X, \mathscr{L})$, the resulting short exact sequence

$$0 \longrightarrow \mathscr{L}^{\otimes -1} \xrightarrow{s} \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0 \tag{9}$$

defines an effective Cartier divisor $D \subset X$. The goal of this section is to understand the finite F-algebra $E = H^0(D, \mathscr{O}_D)$ in dependence of the global section $s \neq 0$. We start with the following observation:

Lemma 5.1. *Suppose $h^1(\mathscr{O}_X) = 0$. Then the number $h^0(\mathscr{O}_D) = [E : F]$ does not depend on the global section $s \neq 0$.*

Proof. The short exact sequence (9) yields a long exact sequence

$$H^0(X, \mathscr{L}^{\otimes -1}) \longrightarrow H^0(X, \mathscr{O}_X) \longrightarrow H^0(D, \mathscr{O}_D) \longrightarrow H^1(X, \mathscr{L}^{\otimes -1}) \longrightarrow H^1(X, \mathscr{O}_X).$$

The term on the right vanishes by assumption, and the term on the left is zero because X is integral and $\mathscr{L} \not\simeq \mathscr{O}_X$ admits a non-zero global section. Thus $h^0(\mathscr{O}_D) = 1 + h^1(\mathscr{L}^{\otimes -1})$, which obviously does not depend on the global section. $\square$

Suppose now we have two non-zero global sections $s_0, s_1 \in H^0(X, \mathscr{L})$ such that the resulting common zero-scheme $Z \subset X$ has codimension two, and is also reduced. Write $D_i \subset X$ with $i = 0, 1$ for the resulting effective Cartier divisors. Each irreducible component of D_i has codimension one, according to Krull's Principal Ideal Theorem. Moreover, there are no common irreducible components, because $Z = D_1 \cap D_2$ has codimension two. Being normal, the scheme X satisfies Serre's Condition (S_2), thus D_i satisfies (S_1), and we conclude that Z is an effective Cartier divisor inside D_i. Thus at each point $z \in Z$, the sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ for the closed subscheme Z is generated by a regular sequence contained in the maximal ideal $\mathfrak{m}_z \subset \mathscr{O}_{X,z}$. For each generic point $\zeta \in Z$, the local ring $\mathscr{O}_{Z,\zeta}$ and hence also $\mathscr{O}_{X,\zeta}$ are regular.

Let $V = \mathrm{Bl}_Z(X)$ be the blowing-up, and write $r : V \to X$ for the resulting morphism. First note that $r_*(\mathscr{O}_V) = \mathscr{O}_X$ and $R^i r_*(\mathscr{O}_V) = 0$ for $i \geq 1$, according to

[4], Exposé VII, Lemma 3.5, and thus $h^j(\mathscr{O}_V) = h^j(\mathscr{O}_X)$ for all $j \geq 0$. According to [30], Proposition 3.4, the scheme V satisfies Serre's Condition (S_2), and furthermore is regular over an open set $U \subset X$ that contains all codimension one-points in X and all generic points in Z. It follows that for each point $\xi \in V$ of codimension one, the local ring $\mathscr{O}_{V,\xi}$ is regular, hence V is normal. Moreover, the exceptional divisor $R = g^{-1}(Z)$ is reduced.

Consider the invertible sheaf $\mathscr{N} = r^*(\mathscr{L})(1) = r^*(\mathscr{L})(-R)$. The short exact sequence $0 \to \mathscr{N} \to r^*(\mathscr{L}) \to r^*(\mathscr{L})|R \to 0$ yields an exact sequence

$$0 \longrightarrow H^0(V, \mathscr{N}) \longrightarrow H^0(V, r^*(\mathscr{L})) \longrightarrow H^0(R, r^*(\mathscr{L})|R).$$

In turn, the $s \in H^0(X, \mathscr{L})$ that vanishes along Z can also be seen as the elements in $H^0(V, \mathscr{N})$. Each such $s \neq 0$ thus defines effective Cartier divisors $D \subset X$ and $D' \subset V$, where the latter is the strict transform of the former, and the induced map $r : D' \to D$ is the blowing-up with center $Z \subset D$. Since Z is already Cartier inside D, this gives an identification $D' = D$.

The strict transforms D_0' and D_1' are disjoint, which follows from [38], Lemma 4.4 and our assumption that Z is reduced. In turn, the invertible sheaf $\mathscr{N}$ is globally generated by s_0 and s_1, which define a morphism

$$h : V \longrightarrow \mathbb{P}^1 \quad \text{with} \quad h^*(\mathscr{O}_{\mathbb{P}^1}(1)) = \mathscr{N}.$$

The morphism is surjective, because $\mathscr{N}$ is numerically non-trivial, and therefore flat. Let $B = \operatorname{Spec} h_*(\mathscr{O}_V)$ be the Stein factorization, with resulting morphisms $f : V \to B$ and $g : B \to \mathbb{P}^1$. Then $\mathscr{O}_B = f_*(\mathscr{O}_V)$, so B is normal, with $h^0(\mathscr{O}_B) = 1$. Moreover, the Leray–Serre spectral sequence gives an exact sequence

$$0 \longrightarrow H^1(B, \mathscr{O}_B) \longrightarrow H^1(V, \mathscr{O}_V) \longrightarrow H^0(B, R^1 f_*(\mathscr{O}_V)) \longrightarrow H^2(B, \mathscr{O}_B), \tag{10}$$

and the term on the right vanishes. The following result reveals that the prime $p = 2$ plays a special role in the general theory of linear systems. It was already used in the previous section, and constitutes our second main result:

Theorem 5.2. *In the above setting, suppose that $H^1(X, \mathscr{O}_X) = 0$, and that for each non-zero linear combination $t = \lambda_0 s_0 + \lambda_1 s_1$, the resulting effective Cartier divisor $D_t \subset X$ is reduced and geometrically connected. The the following holds:*

- (i) *The Stein factorization B is a regular genus-zero curve.*
- (ii) *The map $g : B \to \mathbb{P}^1$ is a universal homeomorphism, with $\deg(g) \mid 2$.*
- (iii) *For each non-zero $s \in H^0(X, \mathscr{L})$ the resulting $D \subset X$ has $h^0(\mathscr{O}_D) = \deg(g)$.*
- (iv) *If $\deg(g) = 2$ then the ground field F is imperfect of characteristic $p = 2$, and B is a twisted line or twisted ribbon.*

Proof. Assertion (i) immediately follows from the exact sequence (10). Given a rational point $t \in \mathbb{P}^1$, we set $T = g^{-1}(t)$ and consider the schematic fiber $V_t = h^{-1}(t) = f^{-1}(T)$. The projection $f : V_t = f^{-1}(T) \to T$ is surjective and flat. Note that V_t is identified with the effective Cartier divisor $D_t \subset X$ defined by the global section $t = \lambda_0 s_0 + \lambda_1 s_1$. By assumption, V_t is reduced and geometrically connected, so the same holds for T. In particular, $T = \{b\}$ is a singleton, with coordinate ring $\kappa(b)$. So Lemma 5.3 below gives (ii) and (iv).

It remains to verify (iii). Since the morphism $f : V \to B$ is flat, the function $b \mapsto \dim_{\kappa(b)} H^0(V_b, \mathscr{O}_{X_b})$ is upper semicontinuous ([21], Chapter III, Theorem 12.8). So the set $U \subset B$ where it takes the generic value is open. If the residue field of $t = g(b)$ is separable, the inclusion $\operatorname{Spec} \kappa(b) \subset g^{-1}(t)$ is an equality, and hence

$$\dim_{\kappa(b)} H^0(V_b, \mathscr{O}_{V_b}) = \deg(g)^{-1} \cdot \dim_{\kappa(t)} H^0(V_t, \mathscr{O}_{V_t}). \tag{11}$$

Applying Lemma 5.1 with the base-change $X \otimes \kappa(t)$, we see that the right-hand side does not depend on the separable point $t \in \mathbb{P}^1$, and infer that $b \in U$ whenever the image $f(b) \in \mathbb{P}^1$ is separable.

By Grauert's Criterion ([21], Chapter III, Corollary 12.9), the formation of $f_*(\mathscr{O}_X)$ commutes with base-change over U. Thus $H^0(X_b, \mathscr{O}_{X_b})$ are one-dimensional vector spaces over $\kappa(b)$ for all $b \in U$. If furthermore $t = f(b)$ is separable we get $h^0(\mathscr{O}_{X_t}) = \deg(g)$ from (11). Now Lemma 5.1 yields (iii). $\square$

The above arguments rest on the following key observation:

Lemma 5.3. *Let B be a regular genus-zero curve, and $g : B \to \mathbb{P}^1$ be a surjective morphism of degree $d \geq 2$. Suppose that for each rational point $t \in \mathbb{P}^1$, the fiber $g^{-1}(t)$ is reduced and geometrically connected. Then the field F is imperfect of characteristic $p = 2$, the curve B is a twisted line or twisted ribbon, and $g : B \to \mathbb{P}^1$ is a universal homeomorphism of degree $d = 2$.*

Proof. Suppose the field F is perfect. Choose a geometric point $\operatorname{Spec}(\Omega) \to \mathbb{P}^1$ over some rational point $t \in \mathbb{P}^1$. The resulting geometric fiber is both reduced and connected, hence isomorphic to $\operatorname{Spec}(\Omega)$, which results in the contradiction $d = 1$. Thus F is imperfect, and we are in characteristic $p > 0$.

Let $K = \mathscr{O}_{\mathbb{P}^1,\eta}$ and $L = \mathscr{O}_{B,\eta}$ be the function fields of our curves. We start with an observation on intermediate fields $K \subsetneqq L' \subset L$. In light of [19], Proposition 7.4.18 and Corollary 7.4.13, each such L' defines a regular curve B', together with a factorization $g = g' \circ h$ into $h : B \to B'$ and $g' : B' \to \mathbb{P}^1$. Note that all three morphisms are surjective, finite and flat. We claim that $B'(F) = \varnothing$. Indeed, if there is a rational point $b' \in B'$, the image $t = g'(b')$ is rational, and the fiber $Z = h^{-1}(b)$ is a finite subscheme with $h^0(\mathscr{O}_Z) = [L' : K] \geq 2$. Since $b \in Z$, the scheme Z is non-reduced or disconnected, in contradiction to our assumption on the fibers of $g : B \to \mathbb{P}^1$.

Applying this observation with $L' = L$, we see that B contains no rational point. Let L' be the relative separable closure of $K \subset L$. Then the generic fiber of $g' : B' \to \mathbb{P}^1$ is étale, so there is a non-empty open set $U \subset \mathbb{P}^1$ over which g' is étale. Since F is infinite, there must be rational points $t \in U$. If $[L' : K] = \deg(B'/\mathbb{P}^1)$ is greater than one, the fiber $Z = g^{-1}(t)$ is geometrically disconnected, contradiction. It follows that the field extension $K \subset L$ is purely inseparable ([11], Chapter V, §7, No. 8, Proposition 13), hence $g : B \to \mathbb{P}^1$ is a universal homeomorphism, of degree $d = p^r$ for some $r \geq 1$. If $p \neq 2$ the invertible sheaf $g^*\mathscr{O}_{\mathbb{P}^1}(1)$ has odd degree, in contradiction to Proposition 1.2. This establishes $p = 2$. From $B(F) = \varnothing$ we also see that B is either a twisted line or a twisted ribbon.

Since $K \subset L$ is purely inseparable, we find a chain of intermediate fields $K = L_0 \subset \ldots \subset L_r = L$ with $[L_{i+1} : L_i] = p$. Consider the corresponding regular curves

B_i and finite surjective morphisms

$$B = B_r \longrightarrow B_{r-1} \longrightarrow \ldots \longrightarrow B_0 = \mathbb{P}^1.$$

It remains to show $r = 1$, and for this it suffices to verify that $B' = B_{r-1}$ is a genus-zero curve containing a rational point. Let $f : B \to B'$ be the given morphism. The short exact sequence

$$0 \longrightarrow \mathscr{O}_{B'} \longrightarrow f_*(\mathscr{O}_B) \longrightarrow \mathscr{L}^{\otimes -1} \longrightarrow 0, \tag{12}$$

defines an invertible sheaf $\mathscr{L}$ on B'. The trace map $f_*(\mathscr{O}_B) \to \mathscr{O}_{B'}$ vanishes on the subsheaf $\mathscr{O}_{B'}$ because $d = p > 0$, and the induced map $\mathscr{L}^{\otimes -1} \to \mathscr{O}_{B'}$ is non-zero since B is reduced. Hence $\deg(\mathscr{L}) \geq 0$. On the other hand, the long exact sequence for (12) reveals

$$h^0(\mathscr{O}_{B'}) = 1 \quad \text{and} \quad h^0(\mathscr{L}^{\otimes -1}) = h^1(\mathscr{O}_{B'}) \quad \text{and} \quad h^1(\mathscr{L}^{\otimes -1}) = 0.$$

This gives $\deg(\mathscr{L}) = \chi(\mathscr{O}_{B'}) - \chi(\mathscr{L}^{\otimes -1}) = 1 - 2h^1(\mathscr{O}_{B'})$. Consequently B' is a genus-zero curve and $\mathscr{L}$ has degree one. It follows that B' contains rational points. $\square$

6. Bend and break

We keep the notation as in Section 3, such that X is a minimal regular surface with invariants $h^0(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = h^0(\omega_X^{\otimes 2}) = 0$, over a ground field F. The third main results of this paper is the following:

Theorem 6.1. *Suppose there is an integral curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and $C^2 = 4$. Then the curve is linearly equivalent to some $C' \subset X$ that is not integral.*

In other words, "bending" the curve C within its linear system, it "breaks" into some $C' = C'_1 + C'_2$. The proof requires extensive preparation, and will be completed in Section 11. Special cases will already treated at the respective ends of this and the following two sections.

Our approach depends on certain maps $f : X \to \mathbb{P}^3$ that we introduce now. First note that $h^0(\mathscr{O}_C) = 1$, according to Proposition 4.1. By Proposition 3.5, the invertible sheaf $\mathscr{L} = \mathscr{O}_X(C)$ is globally generated, the restriction map $H^0(X, \mathscr{L}) \to H^0(C, \mathscr{L}|C)$ is surjective, with $h^0(\mathscr{L}) = 6$ and $h^0(\mathscr{L}|C) = 5$.

We now choose a four-dimensional linear system as follows: Select two sections on $\mathscr{L}|C$ without common zeros, and extend them to global sections $s_1, s_2 \in H^0(X, \mathscr{L})$. Choose one more global section s_3 so that the restrictions $s_1|C, s_2|C, s_3|C$ are linearly independent, and let s_0 be a global section whose zero-scheme is C. This defines a morphism

$$f : X \longrightarrow \mathbb{P}^3 \quad \text{with} \quad f^*\mathscr{O}_{\mathbb{P}^3}(1) = \mathscr{L},$$

having $f^{-1}(t_i) \otimes 1 = s_i$ when writing $\mathbb{P}^3 = \operatorname{Proj} F[t_0, \ldots, t_3]$. The schematic image $V \subset \mathbb{P}^3$ is an integral surface. It is not a linear subscheme, because the canonical map $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathscr{L})$ is injective.

The induced morphism $f : X \to V$ is an alteration. Let $Y = \operatorname{Spec} f_*(\mathscr{O}_X)$ be the Stein factorization, and write

$$h : X \longrightarrow Y \quad \text{and} \quad g : Y \longrightarrow V$$

for the resulting morphisms.

Proposition 6.2. *The finite morphism $g : Y \to V$ is flat over the regular locus* $\mathrm{Reg}(V)$*, and the birational morphism $h : X \to Y$ is the minimal resolution of singularities for the normal surface Y.*

Proof. The scheme Y is Cohen–Macaulay, and over the complement of $\mathrm{Sing}(V)$ the finite surjective morphisms $g : Y \to V$ must be flat ([45], page IV-37, Proposition 22). The second statement holds because the regular surface X is minimal. $\square$

To simplify notation, the pullbacks of the invertible sheaf $\mathscr{O}_{\mathbb{P}^3}(1)$ are also written as $\mathscr{O}_V(1)$ and $\mathscr{O}_Y(1)$ and $\mathscr{O}_X(1) = \mathscr{L}$.

Proposition 6.3. *The scheme V is Gorenstein with $\omega_V = \mathscr{O}_V(d-4)$, where $d =$* $\deg(V)$*, and the numerical invariants are $h^0(\mathscr{O}_V) = 1$ and $h^1(\mathscr{O}_V) = 0$. Moreover, the canonical maps*

$$H^0(\mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(\mathscr{O}_V(1)) \quad \textit{and} \quad H^0(\mathscr{O}_Y(1)) \to H^0(\mathscr{O}_X(1)) = H^0(X, \mathscr{L}) \tag{13}$$

are bijective, and $H^0(\mathscr{O}_V(1)) \to H^0(\mathscr{O}_Y(1))$ is injective.

Proof. Since $V \subset \mathbb{P}^3$ is an effective Cartier divisor, it must be Gorenstein, and the Adjunction Formula gives $\omega_V = \mathscr{O}_V(d-4)$. From the long exact sequence stemming from $0 \to \mathscr{O}_{\mathbb{P}^3}(-d) \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_V \to 0$, we immediately get the numerical invariants.

From the long exact sequence for $0 \to \mathscr{O}_{\mathbb{P}^3}(1-d) \to \mathscr{O}_{\mathbb{P}^3}(1) \to \mathscr{O}_V(1) \to 0$ and the vanishing of $H^i(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1-d))$ for $i \leq 1$ one sees that the restriction map $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(V, \mathscr{O}_V(1))$ is bijective. We have $h_*(\mathscr{O}_X) = \mathscr{O}_Y$, so the Projection Formula ensures that $H^0(\mathscr{O}_Y(1)) \to H^0(\mathscr{O}_X(1))$ is bijective. By construction, the composite map $H^0(\mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathscr{L})$ is injective, hence the same holds for $H^0(\mathscr{O}_V(1)) \to H^0(\mathscr{O}_Y(1))$. $\square$

From (7) we see that there are two cases: Either $V \subset \mathbb{P}^3$ is a quadric surface and $g : Y \to X$ is a double covering, or $V \subset \mathbb{P}^3$ is a quartic surface and $g : Y \to V$ is birational. The former case, together with other special situations, can be treated quickly:

Proposition 6.4. *Theorem 6.1 holds under any of the following conditions:*

(i) *There is a closed point $x \in X$ of degree at least five mapping to a point $v \in V$ of degree at most two.*
(ii) *The singular locus of Y contains a point whose image on V has degree at most three.*
(iii) *The normal surface Y contains a closed point of degree three.*
(iv) *The surface $V \subset \mathbb{P}^3$ has degree two.*

Proof. (i) Suppose some closed point $x \in X$ has $[\kappa(x) : F] \geq 5$ and maps to a point $v \in V$ with $[\kappa(v) : F] \leq 2$. Then there are two planes $H_1 \neq H_2$ with $v \in H_1 \cap H_2$. Set $C_i = f^{-1}(D_i)$. We are done one of the C_i non-integral, so we assume that both are integral. Then $C_1 \cap C_2$ is zero-dimensional, with $h^0(\mathscr{O}_{C_1 \cap C_2}) = (C_1 \cdot C_2) = C^2 = 4$. By construction $x \in C_1 \cap C_2$, giving $4 \geq h^0(\mathscr{O}_{C_1 \cap C_2, x}) \geq 5$, contradiction.

(ii) Suppose some $y \in \mathrm{Sing}(Y)$ has an image $v = g(y)$ with $[\kappa(v) : F] \leq 3$. Using $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) = 4$, we find some plane $H \subset \mathbb{P}^3$ containing v. Then $C' = f^{-1}(H)$ is a curve linearly equivalent to $C \subset X$. It contains the strict transform of the curve $H \cap V$, together with some exceptional divisor, hence is reducible.

(iii) Now we have a closed point $y \in Y$ of degree three. In light of the previous cases, we merely have to treat the case that the local ring $\mathscr{O}_{Y,y}$ is regular. Again we find a plane $H \subset \mathbb{P}^3$ containing v. Then $C' = f^{-1}(H)$ is a curve linearly equivalent to C, containing a point of degree three. It is thus a copy of the projective line, and we find two rational points $x \neq x'$ on X. Let $Z \subset X$ be the closed subscheme of degree four with coordinate ring $\mathscr{O}_{X,x}/\mathfrak{m}_x^2 \times \kappa(x')$, and recall that $\mathscr{L} = \mathscr{O}_X(C)$ has $h^0(\mathscr{L}) = C^2 + 2 = 6$. So some curve C' that is linearly equivalent to C contains Z. By Proposition 1.2, this curve is non-integral.

(iv) Now the finite morphism $g : Y \to V$ has degree two. Seeking a contradiction, we assume that every curve $C' \subset X$ linearly equivalent to C is integral. The alteration $f : X \to V$ factors over the normalization $\tilde{V}$, hence $h^0(\mathscr{O}_{\tilde{V}}) \leq h^0(\mathscr{O}_X) = 1$. Proposition 2.3 ensures that V is normal, and that the singular locus contains at most one point. For each plane $H \subset \mathbb{P}^3$, the intersection $D = V \cap H$ and its preimage $C' = f^{-1}(D)$ are integral genus-zero curves. The latter is linearly equivalent to C, and the morphism $f : C' \to D$ has degree two.

Now choose H disjoint from $\mathrm{Sing}(V)$. Then $f : C' \to D$ is flat, and we see from Proposition 1.6 that D is isomorphic to $\mathbb{P}^1$. It follows that the regular locus of V contains rational points. Fix such a rational point $v \in V$ and set $Z = \mathrm{Spec}(\mathscr{O}_{V,v}/\mathfrak{m}^2)$. Using $h^0(\mathscr{O}_Z) = 3$ and $h^0(\mathscr{O}_{\mathbb{P}^3}(1)) = 4$ we can find another plane $H \subset \mathbb{P}^3$, now containing Z. This gives new integral genus-zero curves $D = V \cap H$ and $C' = f^{-1}(D)$, related by a morphism $f : C' \to D$ of degree two. By construction, the local ring $\mathscr{O}_{D,v}$ is singular. According to Lemma 2.3, our plane H must be disjoint from $\mathrm{Sing}(V)$, and it follows that $f : C' \to D$ is flat. As above we get $D = \mathbb{P}^1$. But this contradicts that D is singular. $\square$

7. Non-normal quartic surfaces

We keep all assumptions of the preceding section, and continue to work towards the proof for Theorem 6.1. Recall that X is a minimal regular surface over the ground field F with invariants $h^0(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = h^0(\omega_X^{\otimes 2}) = 0$, containing an integral curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and $C^2 = 4$. *We now additionally assume that our linear system maps X to a quartic surface $V \subset \mathbb{P}^3$.* Then $f : X \to V$ is birational, $g : Y \to V$ is the normalization, and $h : X \to Y$ is the minimal resolution of singularities.

Proposition 7.1. *The quartic surface $V \subset \mathbb{P}^3$ is non-normal.*

Proof. Suppose V is normal, such that $Y = V$. Form the transcendental field extension $F' = F(t)$. One easily checks that $X \otimes F'$ remains regular and minimal, and $C \otimes F'$ stays integral, and V is normal if and only if this holds for $V \otimes F'$. So without loss of generality we may assume that F is infinite. We then find a plane $H \subset \mathbb{P}^3$ that avoids the finite set $\mathrm{Sing}(V)$. Then the projection of $C' = f^{-1}(D)$ to $D = H \cap V$ is an isomorphism. By the Genus Formula, the quartic plane curve $D \subset H$ has invariants $h^0(\mathscr{O}_D) = 1$ and $h^1(\mathscr{O}_D) = (4-1)(4-2)/2 = 3$. On the other hand, $C' \subset X$ is linearly equivalent to C, hence has $h^1(\mathscr{O}_{C'}) = 0$, contradiction. $\square$

The normalization map $g : Y \to V$ comes with a *branch curve* $B \subset V$, defined as the schematic support of $g_*(\mathscr{O}_Y)/\mathscr{O}_V$. Its schematic preimage $R = g^{-1}(B)$ is called the *ramification curve*. Both are indeed equidimensional of dimension one, and without embedded components, because Y and V are Cohen–Macaulay. Relative Duality for the normalization map gives the formula $g_*(\omega_{Y/V}) = \underline{\mathrm{Hom}}(g_*(\mathscr{O}_Y), \mathscr{O}_V)$. The latter coincides with $g_*(\mathscr{O}_Y(-R))$, and we obtain

$$\omega_Y = \omega_{Y/V} \otimes f^*(\omega_V) = \omega_Y = \mathscr{O}_Y(-R). \tag{14}$$

For more details we refer to [15], Appendix A. The situation is summarized in the commutative diagram

$$\begin{tikzcd} & X \arrow[d, "h"'] \arrow[dd, bend left, "f"] & \\ R \arrow[r] \arrow[d] & Y \arrow[d, "g"'] & \\ B \arrow[r] & V \arrow[r] & \mathbb{P}^3, \end{tikzcd} \tag{15}$$

all horizontal arrows are closed embeddings. Also note that the annihilator ideal for $g_*(\mathscr{O}_Y)/\mathscr{O}_V$ is called the *conductor ideal* for $g : Y \to V$, and accordingly the lower left part of the diagram is called the *conductor square*.

In our situation, the branch curve B can also be seen as a *space curve* in $\mathbb{P}^3$, which will be crucial throughout. Recall that each space curve $Z \subset \mathbb{P}^3$ has some *degree*, defined as

$$\deg(Z) = \chi(\mathscr{O}_Z(1)) - \chi(\mathscr{O}_Z) = \deg(\mathscr{O}_Z(1)) = (\mathscr{O}_{\mathbb{P}^3}(1) \cdot C) \geq 1.$$

Space curves Z of degree one are called *lines*. They are integral, with $\mathscr{O}_Z(1)$ globally generated of degree one. This sheaf is then generated by two global sections, and the resulting morphism $Z \to \mathbb{P}^1$ of degree one is an isomorphism. One finds that the lines are precisely the intersections of two different planes. Note that space curves Z contained in a plane are rather special: By the Genus Formula they have invariants $h^0(\mathscr{O}_Z) = 1$ and $h^1(\mathscr{O}_Z) = (d-1)(d-2)/2$, where $d = \deg(Z)$.

Proposition 7.2. *The branch curve $B \subset \mathbb{P}^3$ has degree three. Moreover, B is integral provided that it contains no line.*

Proof. Let $\eta \in B$ be some generic point, and set $\Lambda = \mathscr{O}_{B,\eta}$ and $\Lambda' = f_*(\mathscr{O}_R)_\eta$. According to [15], Proposition A.2 the lengths of these Λ-modules are related by $\mathrm{length}(\Lambda') = 2\,\mathrm{length}(\Lambda)$. In light of [27], Proposition 6 on page 299, this ensures $\deg(\mathscr{N}_R) = 2\deg(\mathscr{N})$ for every invertible sheaf $\mathscr{N}$ on the branch curve.

Let $E_1, \ldots, E_r$ be the irreducible components of the exceptional divisor $\mathrm{Exc}(X/Y)$, endowed with reduced scheme structure, and $R' \subset X$ be the strict transform of the ramification curve $R \subset Y$. Write

$$K_{X/Y} = -\sum \lambda_i E_i \quad \text{and} \quad f^*(R) = R' + \sum \mu_i E_i$$

for certain $\lambda_i, \mu_i \in \mathbb{Q}$. Then the numerical class of ω_X is given by the $\mathbb{Q}$-divisor $K_{X/Y} - f^*(R)$. With $\mathscr{L} = \mathscr{O}_V(1)$ we compute

$$(\omega_X \cdot C) = (K_{X/Y} - f^*(R)) \cdot f^*(\mathscr{L}) = -R \cdot g^*(\mathscr{L}) = -2\deg(\mathscr{L}|C) = -2\deg(B),$$

using the Projection Formulas. (In the above arguments we have use Mumford's rational pullbacks and rational intersection numbers; for details see the discussion in [34] and [44]). On the other hand, the Adjunction Formula gives

$$(\omega_X \cdot C) = \deg(\omega_C) - C^2 = -2 - 4 = -6.$$

Combining the above equations yields $\deg(B) = 3$.

Suppose that B contains no line. Let $\eta_1, \ldots, \eta_r \in B$ be the generic points and $B_1, \ldots, B_r \subset B$ be the corresponding irreducible components, viewed as the schematic closure of the canonical morphism $\operatorname{Spec}(\mathscr{O}_{B,\eta_i}) \to B$. Then

$$3 = \deg(B) = \sum_{i=1}^{r} \deg(B_i) = \sum_{i=1}^{r} \deg(B_{i,\text{red}}) \cdot \operatorname{length}(\mathscr{O}_{B_i,\eta_i}) \geq 2lr,$$

where $l \geq 1$ is the smallest among the $\operatorname{length}(\mathscr{O}_{B_i,\eta_i})$. We see $r = 1$ and $l = 1$, hence B is integral. □

The presence of lines is of little interest:

Proposition 7.3. *Theorem 6.1 holds if the branch curve B contains a line.*

Proof. Choose a line $L \subset B$, and two planes $H_1 \neq H_2$ inside $\mathbb{P}^3$ containing the line. If $H_i \cap V$ is reducible, the same holds for the preimage $C_i = f^{-1}(H_i \cap C)$. We thus may assume that $H_1 \cap V$ and $H_2 \cap V$ have the same support, namely L. Then the preimages $f^{-1}(H_i \cap V)$ are effective Cartier divisors on some normal scheme that are linearly equivalent and have the same support. This is only possible if $f^{-1}(H_1 \cap V) = f^{-1}(H_2 \cap V)$. But this contradicts the injectivity of the map $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathscr{L})$, where $\mathscr{L} = \mathscr{O}_X(C)$. □

8. Twisted cubics and exotic cubics

We keep the assumptions from the preceding section, and continue work towards the proof for Theorem 6.1. Recall that our regular surface X over the ground field F contains an integral curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and $C^2 = 4$, and maps to non-normal quartic surface $V \subset \mathbb{P}^3$, with normalization $g : Y \to V$, and resulting branch curve $B \subset V$ and ramification curve $R \subset Y$. *We now additionally assume that B contains no line, and that $\operatorname{Sing}(Y)$ contains no point whose image on V is rational.*

These two innocuous assumptions are of profound consequence. Among other things, they allow to compute the cohomological invariants of our schemes:

Proposition 8.1. *The normal surface Y has at most rational singularities, and the numerical invariants are $h^0(\mathscr{O}_Y) = 1$ and $h^1(\mathscr{O}_Y) = h^2(\mathscr{O}_Y) = 0$. Moreover, the ramification curve and the branch curve have invariants*

$$h^0(\mathscr{O}_R) = h^1(\mathscr{O}_R) = 1 \quad \text{and} \quad h^0(\mathscr{O}_B) = 1,\ h^1(\mathscr{O}_B) = 0,$$

and the scheme B is integral.

Proof. First note that B is integral by Proposition 7.2. The Leray–Serre spectral sequence for $h : X \to Y$ gives $h^0(\mathscr{O}_Y) = 1$ and $h^1(\mathscr{O}_Y) = 0$, together with an identification $H^2(Y, \mathscr{O}_Y) = H^0(Y, R^1h_*(\mathscr{O}_X))$. The conductor square in (15) yields a short exact sequence $0 \to \mathscr{O}_V \to \mathscr{O}_Y \oplus \mathscr{O}_B \to \mathscr{O}_R \to 0$, and we proceed by examining its long exact sequence. It starts with

$$0 \longrightarrow H^0(\mathscr{O}_V) \longrightarrow H^0(\mathscr{O}_Y) \oplus H^0(\mathscr{O}_B) \longrightarrow H^0(\mathscr{O}_R) \longrightarrow H^1(\mathscr{O}_V).$$

The term on the right vanishes, and we see $h^0(\mathscr{O}_B) = h^0(\mathscr{O}_R)$. Using $h^1(\mathscr{O}_V) = h^1(\mathscr{O}_Y) = 0$, we get another exact sequence

$$0 \longrightarrow H^1(\mathscr{O}_B) \longrightarrow H^1(\mathscr{O}_R) \longrightarrow H^2(\mathscr{O}_V) \longrightarrow H^2(\mathscr{O}_Y) \longrightarrow 0.$$

Suppose that the map on the right is non-zero. It is necessarily bijective, because $h^0(\mathscr{O}_V) = 1$, and thus $h^2(\mathscr{O}_Y) = 1$. In turn, the sheaf $R^1h_*(\mathscr{O}_X)$ must be supported by a rational point $y \in Y$. The image $v \in V$ of this singularity is again a rational point, in contradiction to our standing assumption. Consequently $h^2(\mathscr{O}_Y) = 0$ and $R^1h_*(\mathscr{O}_X) = 0$, such that each singular local ring $\mathscr{O}_{Y,y}$ is a rational singularity, having some non-trivial field extension $F \subset \kappa(y)$. Moreover, we see that $h^1(\mathscr{O}_R) = h^1(\mathscr{O}_B) + 1$.

The short exact sequence $0 \to \mathscr{O}_Y(-R) \to \mathscr{O}_Y \to \mathscr{O}_R \to 0$ gives a long exact sequence

$$H^0(Y, \mathscr{O}_Y) \longrightarrow H^0(R, \mathscr{O}_R) \longrightarrow H^1(Y, \mathscr{O}_Y(-R)).$$

Using $\omega_Y = \mathscr{O}_Y(-R)$ and Serre Duality, we see that the term on the right is is dual to $H^1(Y, \mathscr{O}_Y)$, which vanishes. Thus $h^0(\mathscr{O}_R) = h^0(\mathscr{O}_Y) = 1$. Finally, we have an exact sequence

$$H^1(Y, \mathscr{O}_Y) \longrightarrow H^1(R, \mathscr{O}_R) \longrightarrow H^2(Y, \mathscr{O}_Y(-R)) \longrightarrow H^2(Y, \mathscr{O}_Y).$$

The outer terms vanish, so $h^1(\mathscr{O}_R) = h^2(\mathscr{O}_Y(-R)) = 1$, again by Serre Duality. □

A useful geometric consequence:

Proposition 8.2. *The resolution of singularities $f : X \to V$ factors over the blowing-up $\mathrm{Bl}_B(V) \to V$ with respect to the space curve $B \subset V$.*

Proof. According to Hartshorne [21], Chapter II, Proposition 7.14 the task is verify that $f^{-1}(Z) \subset X$ is an effective Cartier divisor. As $g^{-1}(Z) = R$, it suffices to show that $h^{-1}(R) \subset X$ is an effective Cartier divisor. Since Y has only rational singularities, this indeed holds, according to [42], Proposition 10.5. □

Recall that a space curve $Z \subset \mathbb{P}^3$ of degree three that is isomorphic to the projective line is called a *twisted cubic*. The Adjunction Formula ensures that it is not contained in a plane. Furthermore $\mathscr{O}_Z(1) \simeq \mathscr{O}_{\mathbb{P}^1}(3)$, and the map $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(Z, \mathscr{O}_Z(1))$ is injective, hence bijective. In turn, each twisted cubic can be seen as the third Veronese embedding of $\mathbb{P}^1$. Moreover, the automorphism group $\mathrm{PGL}_4(F)$ acts transitively on the set of twisted cubics. Note that that the classical designation "twisted" refers to the tensor power $\mathscr{O}_{\mathbb{P}^1}(3)$, and has nothing to do with modern usage in connection with twisted forms.

We also have to deal with a more outlandish type of integral space curve of degree three, which are of arithmetic nature and depend on some cubic field extension

$F \subset E$. Use such an extension to form the cocartesian square

$$\begin{array}{ccc} \mathrm{Spec}(E) & \xrightarrow{(1:0)} & \mathbb{P}^1_E \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \longrightarrow & Z. \end{array}$$

Then Z is a genus-zero curve that is non-Gorenstein. One easily checks that $\mathrm{Pic}(Z) \to \mathrm{Pic}(\mathbb{P}^1_E)$ is bijective, and that the invertible sheaf $\mathscr{L}$ with $\mathscr{L}|\mathbb{P}^1_E = \mathscr{O}_{\mathbb{P}^1}(1)$ is very ample, having $\deg(\mathscr{L}) = 3$ and $h^0(\mathscr{L}) = 4$. This yields a closed embedding $Z \subset \mathbb{P}^3$ of degree three, where we suppress the dependence on the cubic field extension from notation. Note that this are cones over any embedding $\mathrm{Spec}(E) \subset \mathbb{P}^2$. For lack of better designation, we call such space curves $Z \subset \mathbb{P}^3$ *exotic cubics*. Since the Hilbert scheme of space curves of degree three and genus zero is irreducible ([33], Theorem 4.1), one may view the exotic cubics as arithmetic degenerations of twisted cubics.

Proposition 8.3. *Let $Z \subset \mathbb{P}^3$ be an integral space curve of degree three, with invariants $h^0(\mathscr{O}_Z) = 1$ and $h^1(\mathscr{O}_Z) = 0$. Then Z is a twisted cubic or an exotic cubic.*

Proof. First note that $\mathscr{O}_Z(1)$ is an invertible sheaf of odd degree. So if the genus-zero curve Z is Gorenstein, it must by isomorphic to the projective line, by Proposition 1.2, and hence is a twisted cubic. Suppose now that Z fails to be Gorenstein. The normalization $\tilde{Z}$ has $h^1(\mathscr{O}_{\tilde{Z}}) = 0$, the pullback $\mathscr{O}_{\tilde{Z}}(1)$ has degree three, and we infer that $\tilde{Z} = \mathbb{P}^1_E$ for some field extension $F \subset E$ whose degree divides three. Let

$$\begin{array}{ccc} \tilde{Z}' & \longrightarrow & \tilde{Z} \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z. \end{array}$$

be the conductor square, and $z_1, \ldots, z_r \in Z$ be the branch points. Set

$$d = [E : F] \quad \text{and} \quad l_i = [\mathscr{O}_{\tilde{Z}',z_i} : E] \quad \text{and} \quad \lambda_i = dl_i - [\mathscr{O}_{Z',z_i} : F].$$

From the short exact sequence $0 \to \mathscr{O}_Z \to \mathscr{O}_{\tilde{Z}} \times \mathscr{O}_{Z'} \to \mathscr{O}_{\tilde{Z}'} \to 0$ we get

$$1 - (d + \sum_{i=1}^{r}(dl_i - \lambda_i)) + d\sum_{i=1}^{r} l_i = 0.$$

This arrangement of terms produces a *partition* $d - 1 = \sum_{i=1}^{r} \lambda_i$. From $r \geq 1$ and $\lambda_i \geq 1$ we infer $d = 3$, and the only possibilities are

$$r = 1,\ \lambda_1 = 2 \quad \text{and} \quad r = 2,\ \lambda_1 = \lambda_2 = 1.$$

Suppose $r = 2$. Then $\mathscr{O}_{Z',z_i} \subset \mathscr{O}_{\tilde{Z}',z_i}$ has F-codimension $\lambda_i = 1$, so the subfield $E_i = E \cap \mathscr{O}_{Z',z_i}$ inside E has F-codimension at most one. From $H^0(Z, \mathscr{O}_Z) = F$ and the exact sequence

$$0 \longrightarrow H^0(Z, \mathscr{O}_Z) \longrightarrow H^0(\tilde{Z}, \mathscr{O}_{\tilde{Z}}) \times H^0(Z', \mathscr{O}_{Z'}) \longrightarrow H^0(\tilde{Z}', \mathscr{O}_{\tilde{Z}'})$$

we infer that at least one inclusion $E_i \subset E$ is strict. However, $[E:F]=3$ precludes such subfields.

Summing up, there is a unique branch point $z \in Z$, and $\mathscr{O}_{Z',z} \subset \mathscr{O}_{\tilde{Z}',z}$ has F-codimension two. From $h^1(\mathscr{O}_Z)=0$ one immediately sees that the preimage of $z \in Z$ is a single point $\tilde{z} \in \tilde{Z}$, and that the residue field extension $\kappa(z) \subset \kappa(\tilde{z})$ is non-trivial. Its F-codimensions is at most two. This codimension equals the F-dimension of $\kappa(z)$-vector space $\kappa(\tilde{z})/\kappa(z)$, hence $d = [\kappa(z):F]$ divides two. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & E & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \kappa(z) & \longrightarrow & \kappa(\tilde{z}) & \longrightarrow & \kappa(\tilde{z})/\kappa(z) & \longrightarrow & 0.
\end{array}$$

The case $d=2$ yields $[\kappa(\tilde{z}):F]=4$, which is also a multiple of $[E:F]=3$, contradiction. Thus $\kappa(z)=F$, and it follows $\kappa(\tilde{z})=E$. The canonical surjection $\mathscr{O}_{\tilde{Z}',z}/\mathscr{O}_{Z',z} \to \kappa(\tilde{z})/\kappa(z) = E/F$ is bijective, because both quotients are two-dimensional F-vector spaces. By the very definition, the conductor ideal $\mathfrak{c} \subset \mathscr{O}_{Z,z}$ is the annihilator ideal of $\mathscr{O}_{\tilde{Z},z}/\mathscr{O}_{Z,z} = \mathscr{O}_{\tilde{Z}',z}/\mathscr{O}_{Z',z}$, which here coincides with $\kappa(\tilde{z})/\kappa(z)$. Consequently $\mathscr{O}_{\tilde{Z}',z} = \kappa(\tilde{z}) = E$ and $\mathscr{O}_{Z',z} = \kappa(z) = F$. In other words, Z is an exotic cubic. □

Proposition 8.4. *Theorem 6.1 holds provided the branch curve B is an exotic cubic.*

Proof. Write $v_0 \in B$ for the unique singularity of the branch curve for the normalization $g: Y \to V$, which is a rational point. The strategy is to verify that the ramification curve $R = g^{-1}(B)$ contains a closed point of degree three, or that all points in $g^{-1}(v_0)$ have degree at least five. Then Proposition 6.4 indeed gives our assertion. To carry out this strategy we proceed as follows: The normalization takes the form $\tilde{B} = \mathbb{P}^1_E$ for some cubic field extension $F \subset E$. If R is not integral, then both maps $\tilde{B} \to B \leftarrow R$ admit sections over the complement of the singularity $v_0 \in B$, and R indeed contains closed points of degree three.

The main task is to treat the case that R is integral. For each $y \in R$ mapping to the singularity $v_0 \in B$ the local ring $\mathscr{O}_{Y,y}$ is regular, by our standing assumption on the map $\mathrm{Sing}(Y) \to V$, and hence $\mathscr{O}_{R,y}$ is Gorenstein. Moreover, the morphism $g: R \to B$ is flat of degree two on the complement of v_0, and we infer that the whole curve R is Gorenstein. Consider the commutative square

$$\begin{array}{ccc}
\tilde{B} & \longleftarrow & \tilde{R} \\
\downarrow & & \downarrow \\
B & \longleftarrow & R
\end{array} \tag{16}$$

where the vertical arrows are the normalizations, and the horizontal morphisms have degree two. The number $h^1(\mathscr{O}_{\tilde{R}})$ is bounded above by $h^1(\mathscr{O}_R)=1$ and is also a multiple of $h^0(\mathscr{O}_{\tilde{B}})=3$, which is only possible when $h^1(\mathscr{O}_{\tilde{R}})=0$. Thus $\tilde{R}$ is a genus-zero curve over some further field extension $\tilde{E} = H^0(\tilde{R}, \mathscr{O}_{\tilde{R}})$, with $E \subset \tilde{E}$ of degree either one or two. We next form the conductor square

$$\begin{array}{ccc} \tilde{R}' & \longrightarrow & \tilde{R} \\ \downarrow & & \downarrow \\ R' & \longrightarrow & R. \end{array}$$

for the normalization $\tilde{R} \to R$. Write $y_1, \ldots, y_r \in R$ for the branch points and set

$$d = [\tilde{E} : F] \quad \text{and} \quad l_i = [\mathscr{O}_{\tilde{R}',y_i} : \tilde{E}] \quad \text{and} \quad \lambda_i = dl_i - [\mathscr{O}_{R',y_i} : F].$$

The short exact sequence $0 \to \mathscr{O}_R \to \mathscr{O}_{\tilde{R}} \times \mathscr{O}_{R'} \to \mathscr{O}_{\tilde{R}'} \to 0$ yields

$$1 - (d + \sum_{i=1}^{r}(dl_i - \lambda_i)) + d\sum_{i=1}^{r} l_i - 1 = 0,$$

which gives a partition $d = \lambda_1 + \ldots + \lambda_r$. We have $\lambda_i = dl_i/2$ because R is Gorenstein ([16], Proposition A.2), and thus get another partition $2 = l_1 + \ldots + l_r$. The only possibilities are

$$r = 1,\, l_1 = 2 \quad \text{and} \quad r = 2,\, l_1 = l_2 = 1.$$

Moreover $d = 3$ or $d = 6$. We now have to go through all possible cases:

Suppose first that $r = 2$. In other words, the normalization produces exactly two branch points $y_1, y_2 \in R$, each having $\mathscr{O}_{\tilde{R}',y_i} = \tilde{E}$. Inside this, $\mathscr{O}_{R',y_i}$ is a subfield of index two. We see $\mathscr{O}_{R',y_i} = \kappa(y_i)$ and the integer d is even, therefore $d = 6$, hence the closed points $y_i \in Y$ have degree three, as desired.

Suppose now $r = 1$. In other words, the normalization comes with exactly one branch point $y_1 \in R$, having $[\mathscr{O}_{\tilde{R}',y_1} : \tilde{E}] = 2$. So the $\tilde{E}$-algebra $\mathscr{O}_{\tilde{R}',y_1}$ is either a quadratic field extension, or the field product $\tilde{E} \times \tilde{E}$, or the ring of dual numbers $\tilde{E}[\epsilon] = \tilde{E} \oplus \epsilon\tilde{E}$. In the first two cases, the subring $\mathscr{O}_{R',y_1}$ is a field, and thus coincides with $\kappa(y_1)$. In the last two cases, the normal curve $\tilde{R}$ contains an $\tilde{E}$-valued point, and is thus isomorphic to the projective line $\mathbb{P}^1_{\tilde{E}}$. So for $d = 3$, we find in all three cases some closed point $y \in R$ of degree three.

Assume now $d = 6$, so the F-algebra $\mathscr{O}_{\tilde{R}',y_1}$ has degree twelve. Suppose first that the subring $\mathscr{O}_{R',y_1}$ is a field. It thus coincides with $\kappa(y_1)$, necessarily with $[\kappa(y_1) : F] = 6$, and thus every closed point $y \in R$ has degree at least six, as desired. We finally come to the most interesting case where $\mathscr{O}_{\tilde{R}',y_1} = \tilde{E}[\epsilon]$ is the ring of dual numbers, and that the subring $\mathscr{O}_{R',y_1}$ is not a field. It takes the form $K + \epsilon U$, for a *field of representatives* K and some non-zero K-vector subspace $\epsilon U \subset \epsilon\tilde{E}$, satisfying

$$6 = [\mathscr{O}_{R',y_1} : F] = [K : F](1 + \dim_K(U)) \quad \text{and} \quad \dim_K(U) = \operatorname{edim}(\mathscr{O}_{R',y_1}).$$

Here $\operatorname{edim}(A) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ denotes the *embedding dimension* of a local noetherian ring $(A, \mathfrak{m}, \kappa)$. By construction, $[K : F]$ is a strict divisor of $[\tilde{E} : F] = 6$. If $[K : F] = 3$ the closed point $y_1 \in Y$ has degree three, as desired. If $[K : F] = 1$ we have $\operatorname{edim}(\mathscr{O}_{Y,y_1}) \geq \operatorname{edim}(\mathscr{O}_{R',y_1}) = 5$. Then y_1 is a rational point in $\operatorname{Sing}(Y)$, in contradiction to our standing assumption that no such point exists.

Finally, we have to rule out the case $[K : F] = 2$ and $\dim_K(U) = 2$. Then the canonical surjection $E \otimes_F K \to \tilde{E}$ is bijective, and we have an identification $\tilde{R} = \tilde{B} \otimes_E \tilde{E} = \tilde{B} \otimes_F K$. Since all closed points $y \in R$ except y_1 have degree at least six, it suffices to treat the case that $g^{-1}(v_0) = \{y_1\}$, in light of Proposition 6.4. Write $\tilde{v}_0 \in \tilde{B}$ and $\tilde{y}_1 \in \tilde{R}$ for the preimages. From (16) we obtain a commutative diagram of cotangent spaces

$$\begin{array}{ccc} \mathfrak{m}_{\tilde{v}_0}/\mathfrak{m}_{\tilde{v}_0}^2 & \longrightarrow & \mathfrak{m}_{\tilde{y}_1}/\mathfrak{m}_{\tilde{y}_1}^2 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow \\ \mathfrak{m}_{v_0}/\mathfrak{m}_{v_0}^2 & \longrightarrow & \mathfrak{m}_{y_1}/\mathfrak{m}_{y_1}^2. \end{array}$$

These vector spaces and linear maps take the explicit form

$$\begin{array}{ccc} \epsilon E & \longrightarrow & \epsilon(E \otimes_F K) \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow \\ \epsilon E & \longrightarrow & \epsilon U. \end{array}$$

Choose an F-basis $\epsilon\alpha_i \in \epsilon E$, $1 \leq i \leq 3$. Their images $\epsilon(\alpha_i \otimes 1)$ in $\epsilon(E \otimes_F K)$ stays K-linearly independent. However, the images in the two-dimensional K-vector space ϵU must be K-linearly dependent, contradiction. $\square$

9. Blowing-ups centered at twisted cubics

We keep the assumptions from the preceding section, and continue work towards the proof for Theorem 6.1. Recall that our regular surface X over the ground field F contains an integral curve $C \subset X$ with $h^1(\mathscr{O}_C) = 0$ and $C^2 = 4$, and maps to non-normal quartic surface $V \subset \mathbb{P}^3$, with normalization $g : Y \to V$, and resulting branch curve $B \subset V$ and ramification curve $R \subset Y$. *We now assume that B is a twisted cubic, and that* $\mathrm{Sing}(Y)$ *contains no point whose image on V is rational.*

Choosing an identification $B = \mathbb{P}^1$ together with $\mathscr{O}_B(1) = \mathscr{O}_{\mathbb{P}^1}(3)$, we see that the inclusion of the twisted cubic is given by $(x_0 : x_1) \mapsto (x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3)$, and its ideal $\mathfrak{a} \subset k[t_0, \ldots, t_3]$ is generated by the three homogeneous polynomials

$$f_0 = t_0t_4 - t_1t_2, \quad \text{and} \quad f_1 = t_0t_3 - t_1^2, \quad \text{and} \quad f_2 = t_1t_3 - t_2^2. \tag{17}$$

Let $P \in k[t_0, \ldots, t_3]$ be a homogeneous polynomial of degree four defining the quartic surface $V \subset \mathbb{P}^3$. The inclusion $B \subset V$ translates into $P \in \mathfrak{a}$. One actually can say much more:

Proposition 9.1. *There is a homogeneous polynomial $\Phi \in F[X_0, X_1, X_2]$ of degree two such that $P = \Phi(f_0, f_1, f_2)$. Moreover, such Φ is irreducible.*

Proof. Since B is reduced and V is singular along it, we must have $P \in \mathfrak{a}^2$. Now write $P = \sum g_{ijd}f_if_j$ for some homogeneous $g_{ijd} \in F[t_0, \ldots, t_3]$ of degree $d \geq 0$. The terms where $\deg(g_{ijd}) + \deg(f_i) + \deg(f_j)$ differs from $\deg(P) = 4$ cancel each other, and discarding them we may assume that each non-zero summand $d = 0$. Summing up, we have $P = \Phi(f_0, f_1, f_2)$ for some ternary quadratic form Φ. The latter must be irreducible, because P is irreducible. $\square$

To proceed, we consider the blowing-up $\mathrm{Bl}_B(\mathbb{P}^3) \to \mathbb{P}^3$ with center the twisted cubic B. The strict transform of V coincides with the blowing-up $\mathrm{Bl}_B(V) \to V$.

According to Proposition 8.2, the morphism $f : X \to V$ factors over $\mathrm{Bl}_B(V)$. Let $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^3}$ be the sheaf of ideals for the twisted cubic. The surjection

$$\mathscr{H} = \left(\bigoplus_{i=0}^{2} \mathscr{O}_{\mathbb{P}^3}\right) \otimes \mathscr{O}_{\mathbb{P}^3}(-2) = \bigoplus_{i=0}^{2} \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{(f_0, f_1, f_2)} \mathscr{I}$$

stemming form (17) defines an inclusion $\mathrm{Bl}_B(\mathbb{P}^3) \subset \mathbb{P}(\mathscr{H}) = \mathbb{P}^3 \times \mathbb{P}^2$. We thus obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dd, "f"'] & & \mathrm{Bl}_B(V) \arrow[ddll] \arrow[d] \arrow[ddrr] & & \\
& & \mathrm{Bl}_B(\mathbb{P}^3) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \\
V \arrow[r] & \mathbb{P}^3 \arrow[rr, dashed, "(f_0:f_1:f_2)"'] & & \mathbb{P}^2 & V_+(\Phi). \arrow[l]
\end{tikzcd} \tag{18}$$

Indeed, the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is defined outside B, and sends $V \smallsetminus B$ to the quadric curve $V_+(\Phi)$. Using that $V \smallsetminus B$ is schematically dense in $\mathrm{Bl}_B(V)$, we immediately see that the induced projection $\mathrm{pr}_2 : \mathrm{Bl}_B(V) \to \mathbb{P}^2$ factors over $V_+(\Phi)$, giving the diagonal arrow to the right.

According to Ray's result ([39], Corollary 3.6) the projection $\mathrm{Bl}_Z(\mathbb{P}^3) \to \mathbb{P}^2$ is isomorphic to the projectivization $\mathbb{P}(\mathscr{E})$, with the locally free sheaf of rank two sitting in a short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}^{\oplus 2}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^2}^{\oplus 4} \longrightarrow \mathscr{E} \longrightarrow 0. \tag{19}$$

Moreover, it follows from loc. cit. Theorem 3.4 that

$$\mathrm{pr}_2^*(\mathscr{O}_{\mathbb{P}^2}(1)) \simeq \mathrm{pr}_1^*(\mathscr{O}_{\mathbb{P}^3}(1)) \otimes \mathscr{O}_{\mathrm{Bl}_B(\mathbb{P}^3)}(-E), \tag{20}$$

where $E = \mathrm{pr}_1^{-1}(B)$ is the exceptional divisors. Note that Ray worked over the complex numbers, but his arguments literally hold true over arbitrary ground fields. In light of the commutative diagram (18), the morphism $X \to \mathrm{Bl}_B(\mathbb{P}^3) = \mathbb{P}(\mathscr{E})$ factors over $\mathbb{P}(\mathscr{E} \mid V_+(\Phi))$.

Theorem 9.2. *In the above situation, the following holds:*

(i) *The integral quadric curve $V_+(\Phi) \subset \mathbb{P}^2$ is regular.*
(ii) *The rank-two locally free sheaf $\mathscr{E}|V_+(\Phi)$ has degree four.*
(iii) *The morphism $X \to \mathbb{P}(\mathscr{E}|V_+(\Phi))$ is an isomorphism.*
(iv) *The curve $C \subset X$ is a section for the projection $X \to V_+(\Phi)$.*

Proof. The projection $X \to V_+(\Phi)$ is surjective, because the fibers for the bundle $\mathrm{pr}_2 : \mathrm{Bl}_B(\mathbb{P}^3) \to \mathbb{P}^2$ are one-dimensional. The projection thus factors over the normalization of $V_+(\Phi)$. This involves a constant field extension if $V_+(\Phi)$ is non-normal, in contradiction to $H^0(X, \mathscr{O}_X) = F$. From the exact sequence (19) we get $\det(\mathscr{E}) = \mathscr{O}_{\mathbb{P}^2}(2)$. Since $V_+(\Phi) \subset \mathbb{P}^2$ has degree two, the restriction $\mathscr{E} \mid V_+(\Phi)$ has degree four. This establishes (i) and (ii).

The morphism $X \to \mathbb{P}(\mathscr{E}|V_+(\Phi))$ is surjective, because $\mathscr{O}_X(C) = f^*(\mathscr{O}_{\mathbb{P}^3}(1))$ and $C^2 > 0$. The scheme $\mathbb{P}(\mathscr{E} \mid V_+(\Phi))$ is regular because the base $V_+(\Phi)$ is regular, and (iii) follows from the minimality of our surface X.

It remains to establish (iv), which is the most interesting part. Our task is to verify that the intersection number $(\mathrm{pr}_2^*(\mathscr{O}_{\mathbb{P}^2}(1)) \cdot C)$ coincides with $(\mathscr{O}_{\mathbb{P}^2}(1) \cdot V_+(\Phi)) = 2$. To achieve this we compute separately with the factors appearing in (20): The first factor contributes $(\mathrm{pr}_1^*(\mathscr{O}_{\mathbb{P}^3}(2)) \cdot C) = 2 \cdot C^2 = 8$. The second factor is the negative of

$$(\mathscr{O}_{\mathrm{Bl}_B(\mathbb{P}^3)}(E) \cdot C) = h^0(\mathscr{O}_{C\cap E}) = h^0(\mathscr{O}_{C\cap X\cap E}) = \deg(\mathscr{O}_{\mathbb{P}^3}(1) \mid R).$$

But $\deg(\mathscr{O}_{\mathbb{P}^3}(1) \mid R) = \deg(R/B) \cdot \deg(B) = 2 \cdot 3 = 6$, which gives the desired $(\mathrm{pr}_2^*(\mathscr{O}_{\mathbb{P}^2}(1)) \cdot C) = 8 - 6 = 2$. □

In particular, X comes with a fibration such that the base and the generic fiber are genus-zero curves. As discussed in Section 11, this actually finalizes the proof for our generalization of Iskoviskih's result in Theorem 3.1. To complete the proof for the bend-and-break statement in Theorem 6.1, it remains to understand the geometry of ruled surfaces over a regular genus-zero curve, which boils down to an analysis of locally free sheaves of rank two on on such curves. We shall carry this out in the next section.

10. Locally free sheaves on genus-zero curves

We now make a digression and study locally free sheaves on regular genus-zero curves. Recall that by Grothendieck's Splitting Theorem, every vector bundle over the Riemann sphere is a sum of line bundles ([17], Theorem 2.1). This carries over to the projective line over any ground field (as in [36], Theorem 2.1.1 or [22], Theorem 4.1). One may rephrase the result by saying that the *indecomposable* locally free sheaves are exactly the invertible sheaves. The situation for elliptic curves over algebraically closed fields is already much more complicated, and was solved by Atiyah [2]. The indecomposable sheaves over twisted lines where determined by Biswas and Nagaray [9], Theorem 4.1 and Novaković [35], Corollary 6.1, using Galois descent.

Fix a ground field F of characteristic $p \geq 0$. Let D be a regular genus-zero curve whose Picard group is generated by the dualizing sheaf, that is, a twisted line or a twisted ribbon. The dualizing sheaf ω_D is invertible of degree $d = -2$, hence $\chi(\omega_D^{\otimes t}) = -2t + 1$. Moreover,

$$h^1(\omega_D^{\otimes t}) = h^0(\omega_D^{\otimes 1-t}) = 0 \quad \text{and} \quad h^0(\omega_D^{\otimes t}) = -2t + 1 \geq 1 \tag{21}$$

provided $t \leq 0$ holds. The vector space $\mathrm{Ext}^1(\mathscr{O}_D, \omega_D) = H^1(C, \omega_D)$ is 1-dimensional, and the resulting non-split extension

$$0 \longrightarrow \omega_D \longrightarrow \mathscr{F}_D \longrightarrow \mathscr{O}_D \longrightarrow 0 \tag{22}$$

defines a locally free sheaf $\mathscr{F}_D$ of rank two with $\det(\mathscr{F}_D) = \omega_D$. Up to isomorphism, it does not depend on the choice of the extension, and is canonically attached to the curve D. We remark in passing that this construction works on every Gorenstein curve without constant field extension. As we shall see, it plays a particularly important role for genus-zero curves.

Note that in the long exact sequence for (22), the connecting map $H^0(D,\mathscr{O}_D)\to H^1(D,\omega_D)$ is non-zero, because the extension is non-split, hence it is bijective, and thus $h^0(\mathscr{F}_D)=h^1(\mathscr{F}_D)=0$.

Proposition 10.1. *The sheaf $\mathscr{F}_D$ is indecomposable, with $\chi(\mathscr{F}_D\otimes\omega_D^{\otimes t})=-4t$. Moreover,*

$$h^1(\mathscr{F}_D\otimes\omega_D^{\otimes t})=0\quad\textit{and}\quad h^0(\mathscr{F}_D\otimes\omega_D^{\otimes t})=-4t\geq 4$$

provided $t\leq -1$ holds.

Proof. Suppose the sheaf is decomposable, and write $\mathscr{F}_D=\omega_D^{\otimes m-2}\oplus\omega_D^{\otimes -m}$ for some integer m. From $h^0(\mathscr{F}_D)=0$ we see $m-2,-m>0$, which gives the contradiction $0>m>2$. Thus $\mathscr{F}_D$ is indecomposable. Recall that for locally free sheaves $\mathscr{E}_i$ of rank r_i and degree d_i we have

$$\deg(\mathscr{E}_1\otimes\mathscr{E}_2)=r_2d_1+r_1d_2\quad\text{and}\quad\chi(\mathscr{E}_i)=d_i+r_i, \tag{23}$$

the latter by Riemann–Roch, and the formula for $\chi(\mathscr{F}_D\otimes\omega_D^{\otimes t})$ follows. Suppose now $t\leq -1$. Then the group $H^1(D,\omega_D^{\otimes t+1})$ vanishes. So tensoring (22) with $\omega_D^{\otimes t}$ yields an exact sequence

$$0\longrightarrow H^0(D,\omega_D^{\otimes t+1})\longrightarrow H^0(D,\mathscr{F}_D\otimes\omega_D^{\otimes t})\longrightarrow H^0(D,\omega_D^{\otimes t})\longrightarrow 0,$$

and the formula for $h^0(\mathscr{F}_D\otimes\omega_D^{\otimes t})$ results from (21). Furthermore, we get an identification $H^1(D,\mathscr{F}_D\otimes\omega_D^{\otimes t})=H^1(D,\omega_D^{\otimes t})$, which indeed vanishes. □

Note that the wedge product $\mathscr{F}_D\otimes\mathscr{F}_D\to\det(\mathscr{F}_D)$ defines identifications

$$\mathscr{F}_D=\mathscr{F}_D^\vee\otimes\omega_D\quad\text{and}\quad\mathscr{F}_D^\vee=\mathscr{F}_D\otimes\omega_D^{\otimes -1},$$

which is a special case of the following general result:

Theorem 10.2. *Up to isomorphism, the indecomposable locally free sheaves on D are the $\omega_D^{\otimes a}$ and $\mathscr{F}_D\otimes\omega_D^{\otimes b}$, with exponents $a,b\in\mathbb{Z}$.*

Proof. We have to show that each locally free sheaf $\mathscr{E}$ of finite rank decomposes into a sum where each summand has the form $\omega_D^{\otimes a}$ or $\mathscr{F}\otimes\omega_D^{\otimes b}$, with various exponents a and b. We proceed by induction on the rank $r\geq 0$. The cases $r=0$ is trivial. Suppose now $r\geq 1$, and that the assertion holds for $r-1$. Replacing $\mathscr{E}$ by a suitable $\mathscr{E}\otimes\omega_D^{\otimes t}$, we may assume that $h^0(\mathscr{E})\neq 0$ but $h^0(\mathscr{E}\otimes\omega_D)=0$. If follows that there is a non-zero $s:\mathscr{O}_D\to\mathscr{E}$, and the resulting cokernel $\mathscr{E}'$ must be locally free. We thus can view $\mathscr{E}$ as an extension

$$0\longrightarrow\mathscr{O}_D\longrightarrow\mathscr{E}\longrightarrow\mathscr{E}'\longrightarrow 0. \tag{24}$$

By the induction hypothesis, $\mathscr{E}'$ is isomorphic to some

$$\mathscr{F}(a_1,\ldots,a_m\mid b_1,\ldots,b_n)=\left(\bigoplus_{i=1}^m\omega_D^{\otimes a_i}\right)\times\left(\bigoplus_{j=1}^n\mathscr{F}_D\otimes\omega_D^{\otimes b_j}\right).$$

Tensoring the short exact sequence (24) with ω_D, we obtain an exact sequence

$$H^0(D,\mathscr{E}\otimes\omega_D)\longrightarrow H^0(D,\mathscr{F}(a_1+1,\ldots\mid b_1+1,\ldots))\longrightarrow H^1(D,\omega_D).$$

The term on the left vanishes, the term on the right is one-dimensional, so the term in the middle is at most one-dimensional. Using Proposition 10.1, we infer that

$b_1+1,\ldots,b_n+1\geq 0$. Arguing in a similar way with (21), we see that $a_i+1\geq 1$ for all $1\leq i\leq m$, with one possible exception $i=s$, which than must have $a_s+1=0$. Summing up, we may assume that

$$b_1,\ldots,b_n,a_1\geq -1 \quad\text{and}\quad a_2,\ldots,a_m\geq 0.$$

For all exponents $t\geq 0$ and $s\geq -1$ the extension groups

$$\operatorname{Ext}^1(\omega_D^{\otimes t},\mathscr{O}_D)=H^1(D,\omega_D^{\otimes -t})\quad\text{and}\quad \operatorname{Ext}^1(\mathscr{F}_D\otimes\omega^{\otimes s},\mathscr{O}_D)=H^1(\mathscr{F}_D\otimes\omega_D^{\otimes -1-s}) \tag{25}$$

vanishes. Consequently, (24) splits provided $a_1\geq 0$ holds, and $\mathscr{E}$ decomposes as desired.

It remains to treat the case $a_1=-1$. The pullback $\mathscr{E}_0=\mathscr{E}\times_{\mathscr{E}'}\omega_D^{\otimes a_1}$ of (24) with respect to the inclusion map $\omega_D^{\otimes a_1}\subset\mathscr{E}'$ sits in a short exact sequence

$$0\longrightarrow\mathscr{O}_D\longrightarrow\mathscr{E}_0\longrightarrow\omega_D^{\otimes -1}\longrightarrow 0. \tag{26}$$

Each such extension is either isomorphic to $\mathscr{F}_D^\vee$ or $\mathscr{O}_D\oplus\omega_D^{\otimes -1}$, because the vector space $\operatorname{Ext}^1(\omega_D^{\otimes -1},\mathscr{O}_D)=H^1(D,\omega_D)$ is one-dimensional. Using the Snake Lemma, we see that our sheaves form a short exact sequence

$$0\longrightarrow\mathscr{E}_0\longrightarrow\mathscr{E}\longrightarrow\mathscr{E}''\longrightarrow 0, \tag{27}$$

now with $\mathscr{E}''=\mathscr{F}(a_2,\ldots,a_m\mid b_1,\ldots,b_n)$. The short exact sequence (26) yields an exact sequence

$$\operatorname{Ext}^1(\mathscr{E}'',\mathscr{O}_D)\longrightarrow\operatorname{Ext}^1(\mathscr{E}'',\mathscr{E}_0)\longrightarrow\operatorname{Ext}^1(\mathscr{E}'',\omega_D^{\otimes -1}).$$

The outer terms vanish, as one sees by using (25) again. In turn, the extension (24) splits, and $\mathscr{E}$ decomposes as desired. □

Proposition 10.3. *The indecomposable sheaves that are globally generated are precisely the $\omega_D^{\otimes a}$ with $a\leq 0$, and $\mathscr{F}_D\otimes\omega_D^{\otimes b}$ with $b\leq -1$. In any case, we have*

$$\deg(\omega_D^{\otimes a})=-2a\quad\textit{and}\quad\deg(\mathscr{F}_D\otimes\omega_D^{\otimes b})=-4b-2.$$

Proof. The degrees immediately follow from (23). Clearly, $\omega_D^{\otimes a}$ is globally generated if and only if $a\leq 0$. If $b\leq -1$, the outer terms in the short exact sequence $0\to\omega_D^{\otimes b+1}\to\mathscr{F}_D\otimes\omega_D^{\otimes b}\to\omega_D^{\otimes b}\to 0$ are globally generated, and $H^1(D,\omega_D^{\otimes b+1})$ vanishes, hence the term in the middle is globally generated. Conversely, if $\mathscr{F}_D\otimes\omega_D^{\otimes b}$ is globally generated, so is the quotient $\omega_D^{\otimes b}$, and hence $b\leq 0$. The case $b=0$ is impossible, since we already observed that $h^0(\mathscr{F}_D)=0$. □

For each locally free sheaf $\mathscr{E}$ of rank two on D yields the regular surface $X=\mathbb{P}(\mathscr{E})$. Each section is of the form $C=\mathbb{P}(\omega_D^{\otimes a})$, for some short exact sequence

$$0\longrightarrow\omega_D^{\otimes a'}\longrightarrow\mathscr{E}\longrightarrow\omega_D^{\otimes a}\longrightarrow 0. \tag{28}$$

The self-intersection is $C^2=\deg(\omega_D^{\otimes a})-\deg(\omega_D^{\otimes a'})=-2(a-a')$, as explained in [16], Lemma 6.1.

Proposition 10.4. *Notation as above. If $\mathscr{E}$ is indecomposable, the selfintersection satisfies $C^2\equiv 2$ modulo 4. If $\mathscr{E}$ is decomposable and $C^2>0$, then $C\subset X$ is linearly equivalent to a curve C' that is reducible.*

Proof. Suppose $\mathscr{E} = \mathscr{F}_D \otimes \omega_D^{\otimes b}$ for some integer b. Taking degrees for (28) we get

$$2 \equiv -4b - 2 = \deg(\mathscr{E}) = -2a - 2a' \equiv -2(a - a') = C^2 \quad \text{modulo } 4,$$

which gives the first assertion. If $\mathscr{E}$ is decomposable we have $\mathscr{E} = \omega_D^{\otimes r} \oplus \omega_D^{\otimes r'}$ with $r + r' = a + a'$. Without loss of generality $r' \leq r$. Then the section $R = \mathbb{P}(\omega_D^{\otimes r})$ has $R^2 = -2(r - r') \leq 0$. Let $F \subset X$ be the fiber over a closed point $b \in B$ of degree two. Then C is numerically equivalent to $R + mF$ for some integer m. From $0 \leq (C \cdot R) = R^2 + m(F \cdot R) = R^2 + 2m$ we get $m \geq -R^2/2 \geq 0$. The case $m = 0$ is impossible, because $C^2 > 0$ and $R^2 \leq 0$, thus $R + mF$ is a reducible curve. It is linearly equivalent to C, because $\mathrm{Pic}^\tau(X) = 0$. $\square$

We close this section with some further observations on sheaves over such D. If D is a twisted line, we have $D \otimes K \simeq \mathbb{P}^1_E$ for some separable quadratic field extension $F \subset K$, with

$$\omega_D \mid \mathbb{P}^1_K = \mathscr{O}_{\mathbb{P}^1_K}(-2) \quad \text{and} \quad \mathscr{F}_D \mid \mathbb{P}^1_K = \mathscr{O}_{\mathbb{P}^1_K}(-1) \oplus \mathscr{O}_{\mathbb{P}^1_K}(-1).$$

If D is a twisted ribbon, we are in characteristic $p = 2$, and $(D \otimes E)_{\mathrm{red}} = \mathbb{P}^1_E$ for some height-one extension $F \subset E$ of degree four. Note that the projection $\mathbb{P}^1_E \to B$ again has degree two, but now

$$\omega_D \mid \mathbb{P}^1_E = \mathscr{O}_{\mathbb{P}^1_E}(-1) \quad \text{and} \quad \mathscr{F}_D \mid \mathbb{P}^1_E = \mathscr{O}_{\mathbb{P}^1_E}(-1) \oplus \mathscr{O}_{\mathbb{P}^1_E}.$$

Also note that the sheaf of Kähler differentials $\Omega^1_{D/k}$ is locally free of rank two. It turns out to be decomposable:

Proposition 10.5. *In the above situation, we have $\Omega^1_{D/k} \simeq \omega_D^{\otimes 2} \oplus \omega_D$.*

Proof. The sheaf $\omega_D^{\otimes -1}$ is very ample and embeds D into $\mathbb{P}^2$ as a curve of degree two. The canonical surjection $\Omega^1_{\mathbb{P}^2/F}|B \to \Omega^1_{D/F}$ of locally free sheaves of rank two must be bijective. Restricting the Euler sequence $0 \to \Omega^1_{\mathbb{P}^2/F} \to \bigoplus_{i=0}^2 \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2} \to 0$ to D yields a short exact sequence

$$0 \longrightarrow \Omega^1_{D/F} \longrightarrow \omega_D^{\oplus 3} \longrightarrow \mathscr{O}_D \longrightarrow 0.$$

This shows $\deg(\Omega^1_{D/F}) = -6$ and $\mathrm{Hom}(\Omega^1_{D/F}, \omega_D) \neq 0$. In light of Theorem 10.2, the only possibilities for $\Omega^1_{D/F}$ are $\mathscr{F}_D \otimes \omega_D$ and $\omega_D^{\otimes 2} \oplus \omega_D$. Note that both have $h^0 = 0$ and $h^1 = 4$, so the numerical invariants do not tell them apart.

Choose a field extension $F \subset E$ such that the base-change $D \otimes E$ becomes the split ribbon

$$\mathbb{P}^1_E \oplus \mathscr{O}_{\mathbb{P}^1_E}(-1) = \mathrm{Spec}\, E[u, \epsilon] \cup \mathrm{Spec}\, E[u^{-1}, \epsilon u^{-1}],$$

where $\epsilon^2 = 0$. Over these affine open sets, the differentials $e_1 = du, e_2 = d\epsilon$ and $e'_1 = du^{-1}$, $e'_2 = d\epsilon u^{-1}$ form a basis for $\Omega^1_{D/F} \otimes E$, and the resulting cocycle takes the form

$$\begin{pmatrix} u^{-2} & \epsilon u^{-2} \\ 0 & u^{-1} \end{pmatrix} \in \mathrm{GL}_2(E[u^{\pm 1}, \epsilon]),$$

as explained in [41], Section 2. With the new basis e'_1, $e'_2 + \epsilon e'_1$ we get

$$e'_1 = u^{-2} e_1 \quad \text{and} \quad e'_2 + \epsilon e'_1 = u^{-1} e_2,$$

and infer that the sheaf $\Omega^1_{D/F}$ and $\omega_D^{\otimes 2}\oplus\omega_D$ become isomorphic after base-change to E. In turn, they are already isomorphic over F. $\square$

Let us finally point out that the classification of locally free sheaves on the split ribbon $D\otimes E=\mathbb{P}^1_E\oplus\mathscr{O}_{\mathbb{P}^1_E}(-1)$ is much more complicated. This can already seen on a base-change $D\otimes K$ that is a denormalization of $\mathbb{P}^1_E$, obtained by replacing an E-valued point $x\in\mathbb{P}^1_E$ by a K-valued point. To give a locally free sheaf on this denormalization amounts to give a sheaf $\mathscr{F}=\mathscr{O}_{\mathbb{P}^1_E}(a_1)\oplus\ldots\oplus\mathscr{O}_{\mathbb{P}^1_E}(a_r)$ on $\mathbb{P}^1_E$, together with an K-rational structure on the E-vector space $V=\mathscr{F}(a)$, that is, an K-subspace V_0 such that the canonical map $V_0\otimes_K E\to V$ is bijective. This apparently involves continuous families. It would be interesting to work this out.

11. Proofs for main results

Recall that X denotes a minimal regular proper surface over a ground field F with invariants $h^0(\mathscr{O}_X)=1$ and $h^1(\mathscr{O}_X)=h^0(\omega_X^{\otimes 2})=0$. We now collect our findings and give the proofs for two main results of this paper.

Proof for Theorem 6.1. Here we have an integral curve $C\subset X$ with $h^1(\mathscr{O}_C)=0$ and $C^2=4$, and the task is to find a linearly equivalent curve C' that is not integral. For this we have introduced in Section 6 a finite morphism $f:X\to\mathbb{P}^3$. Several easy situations where already treated in Proposition 6.4, and it remains to handle the case that the image $V=f(X)$ is an integral quartic surface, which is non-normal by Proposition 7.1. We write $g:Y\to V$ for the normalization, and only have to deal with the case that no singular point $y\in Y$ maps to a rational point $v\in V$, again by Proposition 6.4.

In light of Proposition 7.2 and Proposition 8.4, it remains to treat the case that the branch curve $B\subset V$ for the normalization is a twisted cubic. We then established in Theorem 9.2 that X arises as blowing-up of the quartic surface V with center the twisted cubic B. From this we inferred that $X=\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ of rank two over a regular genus-zero curve D, having C as a section. For $D=\mathbb{P}^1$ the sheaf $\mathscr{E}$ is decomposable, and we find a reducible C' as in the proof for Proposition 10.4. The interesting case is when ω_D generates the Picard group. Recall that $C^2=4$. By Proposition 10.4, the sheaf $\mathscr{E}$ must be decomposable, and C is linearly equivalent to a reducible curve. $\square$

Proof for Theorem 3.1. The task is to show that X is isomorphic to a plane or a quadric surface in $\mathbb{P}^3$, or there is a fibration $f:X\to B$ with $\mathscr{O}_B=f_*(\mathscr{O}_X)$ where the base and the generic fiber are genus-zero curves, or the dualizing sheaf ω_X generates the Picard group $\mathrm{Pic}(X)$. Suppose the dualizing sheaf does not generate. According to Proposition 3.4 there is an integral curve $C\subset X$ with $h^1(\mathscr{O}_C)=0$ and $C^2\geq 0$, such that every linearly equivalent curve C' is integral. In light of Proposition 3.6 and Proposition 4.2 it remains to treat the case where $C^2\geq 3$ and $C^2\mid 4$, in other words $C^2=4$. But then Theorem 6.1 tells us that some C' is non-integral, contradiction. $\square$

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